39. A. $\frac{d}{dx}\sec^{-1}(x^2)$

$$y = \sec^{-1} u \qquad u = x^2$$
$$\frac{d}{du}(\sec y) = \frac{d}{du}u \qquad du = 2x$$
$$\sec y \tan y \frac{dy}{du} = 1$$
$$\frac{dy}{du} = \frac{1}{\sec y \tan y}$$
$$\frac{dy}{du} = \frac{1}{4\sqrt{u^2-1}}\frac{du}{dx}$$
$$= \frac{2x}{x^2\sqrt{x^4-1}}$$
$$= \frac{2}{x\sqrt{x^4-1}}$$

40. C. $\frac{d}{dx}\tan^{-1}(2x)$

See problem 38.

$$u = 2x$$
$$\frac{du}{dx} = 2$$
$$\frac{1}{1+u^2}\frac{du}{dx} = \frac{2}{1+4x^2}$$
$$\frac{2}{1+4(1)^2} = \frac{2}{5}$$

41. **(a)** $y = \frac{\pi}{2}$

(b) $y = -\frac{\pi}{2}$

(c) None, since $\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2} \neq 0.$

42. **(a)** $y = 0$

(b) $y = \pi$

(c) None, since $\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2} \neq 0.$

43. **(a)** $y = \frac{\pi}{2}$

(b) $y = \frac{\pi}{2}$

(c) None, since $\frac{d}{dx}\sec^{-1} x = \frac{1}{|x|\sqrt{x^2-1}} \neq 0.$

44. **(a)** $y = 0$

(b) $y = 0$

(c) None, since $\frac{d}{dx}\csc^{-1} x = -\frac{1}{|x|\sqrt{x^2-1}} \neq 0.$

45. **(a)** None, since $\sin^{-1} x$ is undefined for $x > 1$.

(b) None, since $\sin^{-1} x$ is undefined for $x < -1$.

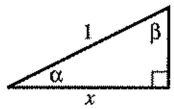
(c) None, since $\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}} \neq 0.$

46. **(a)** None, since $\cos^{-1} x$ is undefined for $x > 1$.

(b) None, since $\cos^{-1} x$ is undefined for $x < -1$.

(c) None, since $\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}} \neq 0.$

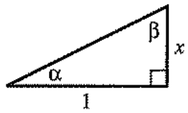
47. **(a)**



$\alpha = \cos^{-1} x, \beta = \sin^{-1} x$

So $\cos^{-1} x + \sin^{-1} x = \alpha + \beta = \frac{\pi}{2}.$

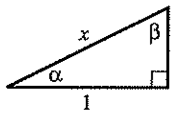
(b)



$\alpha = \tan^{-1} x, \beta = \cot^{-1} x$

So $\tan^{-1} x + \cot^{-1} x = \alpha + \beta = \frac{\pi}{2}.$

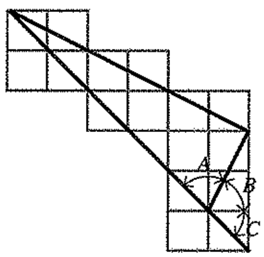
(c)



$\alpha = \sec^{-1} x, \beta = \csc^{-1} x$

So $\sec^{-1} x + \csc^{-1} x = \alpha + \beta = \frac{\pi}{2}.$

48.



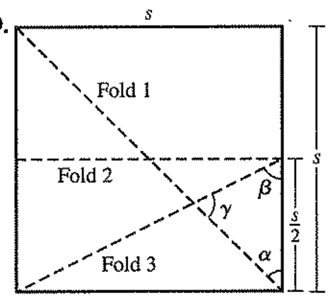
The "straight angle" with the arrows in it is the sum of the three angles *A*, *B*, and *C*.

A is equal to $\tan^{-1} 3$ since the opposite side is 3 times as long as the adjacent side.

B is equal to $\tan^{-1} 2$ since the side opposite it is 2 units and the adjacent side is one unit.

C is equal to $\tan^{-1} 1$ since both the opposite and adjacent sides are one unit long.

But the sum of these three angles is the "straight angle," which has measure π radians.

49. 

If s is the length of a side of the square, then

$\tan\alpha = \frac{s}{s} = 1$, so $\alpha = \tan^{-1} 1$ and

$\tan\beta = \frac{s}{\frac{s}{2}} = 2$, so $\beta = \tan^{-1} 2$.

From Exercise 34, we have

$\gamma = \pi - \alpha - \beta = \pi - \tan^{-1} 1 - \tan^{-1} 2 = \tan^{-1} 3.$

Section 3.9 Derivatives of Exponential and Logarithmic Functions (pp. 172–180)

Exploration 1 Leaving Milk on the Counter

1. The temperature of the refrigerator is 42°F, the temperature of the milk at time $t = 0$.

2. The temperature of the room is 72°F, the limit to which y tends as t increases.

3. The milk is warming up the fastest at $t = 0$. The second derivative $y'' = -30(\ln(0.98))^2(0.98)^t$ is negative, so y' (the rate at which the milk is warming) is maximized at the lowest value of t.

4. We set $y = 55$ and solve;

$$72 - 30(0.98)^t = 55$$
$$(0.98)^t = \frac{17}{30}$$
$$t\ln(0.98) = \ln\left(\frac{17}{30}\right)$$
$$t = \frac{\ln\left(\frac{17}{30}\right)}{\ln(0.98)} = 28.114$$

The milk reaches a temperature of 55° F after about 28 minutes.

5. $\frac{dy}{dx} = -30\ln(0.98)\cdot(0.98)^t$. At $t = \frac{\ln\left(\frac{17}{30}\right)}{\ln(0.98)}$,

$\frac{dy}{dx} \approx 0.343$ degrees/minute.

Quick Review 3.9

1. $\log_5 8 = \frac{\ln 8}{\ln 5}$

2. $7^x = e^{\ln 7^x} = e^{x\ln 7}$

3. $\ln(e^{\tan x}) = \tan x$

4. $\ln(x^2-4) - \ln(x+2) = \ln\frac{x^2-4}{x+2}$
$= \ln\frac{(x+2)(x-2)}{x+2} = \ln(x-2)$

5. $\log_2(8^{x-5}) = \log_2(2^3)^{x-5} = \log_2 2^{3x-15} = 3x-15$

6. $\frac{\log_4 x^{15}}{\log_4 x^{12}} = \frac{15\log_4 x}{12\log_4 x} = \frac{15}{12} = \frac{5}{4}, x > 0$

7. $3\ln x - \ln 3x + \ln(12x^2) = \ln x^3 - \ln 3x + \ln(12x^2)$
$= \ln\frac{(x^3)(12x^2)}{3x} = \ln(4x^4)$

8.
$$3^x = 19$$
$$\ln 3^x = \ln 19$$
$$x\ln 3 = \ln 19$$
$$x = \frac{\ln 19}{\ln 3} \approx 2.68$$

9.
$$5^t\ln 5 = 18$$
$$5^t = \frac{18}{\ln 5}$$
$$\ln 5^t = \ln\frac{18}{\ln 5}$$
$$t\ln 5 = \ln 18 - \ln(\ln 5)$$
$$t = \frac{\ln 18 - \ln(\ln 5)}{\ln 5} \approx 1.50$$

10.
$$3^{x+1} = 2x$$
$$\ln 3^{x+1} = \ln 2^x$$
$$(x+1)\ln 3 = x\ln 2$$
$$x(\ln 3 - \ln 2) = -\ln 3$$
$$x = \frac{\ln 3}{\ln 2 - \ln 3} \approx -2.71$$

Section 3.9 Exercises

1. $\frac{dy}{dx} = \frac{d}{dx}(2e^x) = 2e^x$

2. $\frac{dy}{dx} = \frac{d}{dx}(e^{2x}) = e^{2x}\frac{d}{dx}(2x) = 2e^{2x}$

3. $\frac{dy}{dx} = \frac{d}{dx}e^{-x} = e^{-x}\frac{d}{dx}(-x) = -e^{-x}$

4. $\frac{dy}{dx} = \frac{d}{dx}e^{-5x} = e^{-5x}\frac{d}{dx}(-5x) = -5e^{-5x}$

5. $\frac{dy}{dx} = \frac{d}{dx}e^{2x/3} = e^{2x/3}\frac{d}{dx}\left(\frac{2x}{3}\right) = \frac{2}{3}e^{2x/3}$

6. $\frac{dy}{dx} = \frac{d}{dx}e^{-x/4} = e^{-x/4}\frac{d}{dx}\left(-\frac{x}{4}\right) = -\frac{1}{4}e^{-x/4}$

7. $\frac{dy}{dx} = \frac{d}{dx}(xe^2) - \frac{d}{dx}(e^x) = e^2 - e^x$

8. $\frac{dy}{dx}=\frac{d}{dx}(x^2e^x)-\frac{d}{dx}(xe^x)$
$=(x^2)(e^x)+(e^x)(2x)-[(x)(e^x)+(e^x)(1)]$
$=x^2e^x+xe^x-e^x$

9. $\frac{dy}{dx}=\frac{d}{dx}e^{\sqrt{x}}=e^{\sqrt{x}}\frac{d}{dx}(\sqrt{x})=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$

10. $\frac{dy}{dx}=\frac{d}{dx}e^{(x^2)}=e^{(x^2)}\frac{d}{dx}(x^2)=2xe^{(x^2)}$

11. $\frac{dy}{dx}=\frac{d}{dx}8^x=8^x\ln 8$

12. $\frac{dy}{dx}=\frac{d}{dx}9^{-x}=9^{-x}(\ln 9)\frac{d}{dx}(-x)=-9^{-x}\ln 9$

13. $\frac{dy}{dx}=\frac{d}{dx}3^{\csc x}=3^{\csc x}(\ln 3)\frac{d}{dx}(\csc x)$
$=3^{\csc x}(\ln 3)(-\csc x\cot x)$
$=-3^{\csc x}(\ln 3)(\csc x\cot x)$

14. $\frac{dy}{dx}=\frac{d}{dx}3^{\cot x}=3^{\cot x}(\ln 3)\frac{d}{dx}(\cot x)$
$=3^{\cot x}(\ln 3)(-\csc^2 x)$
$=-3^{\cot x}(\ln 3)(\csc^2 x)$

15. $\frac{dy}{dx}=\frac{d}{dx}\ln(x^2)=\frac{1}{x^2}\frac{d}{dx}(x^2)=\frac{1}{x^2}(2x)=\frac{2}{x}$

16. $\frac{dy}{dx}=\frac{d}{dx}(\ln x)^2=2\ln x\frac{d}{dx}(\ln x)=\frac{2\ln x}{x}$

17. $\frac{dy}{dx}=\frac{d}{dx}\ln(x^{-1})=\frac{d}{dx}(-\ln x)=-\frac{1}{x},\ x>0$

18. $\frac{dy}{dx}=\frac{d}{dx}\ln\frac{10}{x}=\frac{d}{dx}(\ln 10-\ln x)=0-\frac{1}{x}$
$=-\frac{1}{x}, x>0$

19. $\frac{d}{dx}\ln(\ln x)=\frac{1}{\ln x}\frac{d}{dx}\ln x=\frac{1}{\ln x}\cdot\frac{1}{x}=\frac{1}{x\ln x}$

20. $\frac{dy}{dx}=\frac{d}{dx}(x\ln x-x)=(x)\left(\frac{1}{x}\right)+(\ln x)(1)-1$
$=1+\ln x-1=\ln x$

21. $\frac{dy}{dx}=\frac{d}{dx}(\log_4 x^2)=\frac{d}{dx}\frac{\ln x^2}{\ln 4}=\frac{d}{dx}\left[\left(\frac{2}{\ln 4}\right)(\ln x)\right]$
$=\frac{2}{\ln 4}\cdot\frac{1}{x}=\frac{2}{x\ln 4}=\frac{1}{x\ln 2}$

22. $\frac{dy}{dx}=\frac{d}{dx}(\log_5\sqrt{x})=\frac{d}{dx}\frac{\ln x^{1/2}}{\ln 5}=\frac{d}{dx}\frac{\frac{1}{2}\ln x}{\ln 5}$
$=\frac{1}{2\ln 5}\frac{d}{dx}(\ln x)=\frac{1}{2\ln 5}\cdot\frac{1}{x}=\frac{1}{2x\ln 5},\ x>0$

23. $\frac{dy}{dx}=\frac{d}{dx}\log_2\left(\frac{1}{x}\right)=\frac{d}{dx}(-\log_2 x)=-\frac{1}{x\ln 2}, x>0$

24. $\frac{dy}{dx}=\frac{d}{dx}\frac{1}{\log_2 x}=-\frac{1}{(\log_2 x)^2}\frac{d}{dx}(\log_2 x)$
$=-\frac{1}{(\log_2 x)^2}\frac{1}{x\ln 2}=-\frac{1}{x(\ln 2)(\log_2 x)^2}$
or $-\frac{\ln 2}{x(\ln x)^2}$

25. $\frac{dy}{dx}=\frac{d}{dx}(\ln 2\cdot\log_2 x)=(\ln 2)\frac{d}{dx}(\log_2 x)$
$=(\ln 2)\left(\frac{1}{x\ln 2}\right)=\frac{1}{x}, x>0$

26. $\frac{dy}{dx}=\frac{d}{dx}\log_3(1+x\ln 3)$
$=\frac{1}{(1+x\ln 3)\ln 3}\frac{d}{dx}(1+x\ln 3)$
$=\frac{\ln 3}{(1+x\ln 3)\ln 3}=\frac{1}{1+x\ln 3},\ x>-\frac{1}{\ln 3}$

27. $\frac{dy}{dx}=\frac{d}{dx}(\log_{10}e^x)=\frac{d}{dx}(x\log_{10}e)=\log_{10}e=\frac{\ln e}{\ln 10}$
$=\frac{1}{\ln 10}$

28. $\frac{dy}{dx}=\frac{d}{dx}\ln 10^x=\frac{d}{dx}(x\ln 10)=\ln 10$

29. $m=5$
$y=3^x+1$
$y'=3^x\ln 3=5$
$x=1.379$
$y=3^{1.379}+1=5.551$
$(1.379,\ 5.551)$

30. $m_2=-\frac{1}{m_1}=\frac{1}{3}$
$\frac{d}{dx}(2e^{x-1})=2e^x$
$\frac{1}{3}=2e^x$
$\frac{1}{6}=e^x$
$x=-1.792$
$y=2e^{x-1}$
$y=\frac{2}{6}-1=-0.667$
$(-1.792,\ -0.667)$

31. $y=\ln 2x$
$e^y=2x$
$\frac{d}{dx}(e^y)=\frac{d}{dx}(2x)$
$e^y\frac{dy}{dx}=2$
$\frac{dy}{dx}=2e^{-1}$

32. $y = \ln(x/3)$

$\frac{d}{dx}(e^y) = \frac{d}{dx}\left(\frac{x}{3}\right)$

$e^y \frac{dy}{dx} = \frac{1}{3}$

$\frac{dy}{dx} = \frac{1}{3}e^{-1}$

33. $\frac{dy}{dx} = \frac{d}{dx}(x^{\pi}) = \pi x^{\pi-1}$

34. $\frac{dy}{dx} = \frac{d}{dx}(x^{1+\sqrt{2}}) = (1+\sqrt{2})x^{1+\sqrt{2}-1} = (1+\sqrt{2})x^{\sqrt{2}}$

35. $\frac{dy}{dx} = \frac{d}{dx}x^{-\sqrt{2}} = -\sqrt{2x}^{-\sqrt{2}-1}$

36. $\frac{dy}{dx} = \frac{d}{dx}x^{1-e} = (1-e)x^{1-e-1} - (1-e)x^{-e}$

37. $\frac{d}{dx}\ln(x+2) = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}\ln(u) \qquad u = x+2$

$f'(x) = \frac{1}{x+2} \qquad \frac{du}{dx} = 1$

$x+2 > 0$

$x > -2$

38. $\frac{d}{dx}\ln(x+2) = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}\ln(u) \qquad u = 2x+2$

$\frac{du}{dx} = 2$

$= 0 + \frac{d}{dx}\ln x = \frac{1}{x}$

$x+1 > 0$

$x > -1$

39. $\frac{d}{dx}\ln(2-\cos x) = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}\ln(u) \qquad u = 2-\cos x$

$\frac{du}{dx} = \sin x$

$f'(x) = \frac{\sin x}{2-\cos x}$

Domain of x is all real numbers.

40. $\frac{d}{dx}\ln(x^2+1) = \frac{1}{u}\frac{du}{dx}$

$\frac{d}{dx}\ln(u) \qquad u = x^2+1$

$\frac{du}{dx} = 2x$

$f'(x) = \frac{2x}{x^2+1}$

$x^2+1 > 0$, Domain of x is all real numbers.

41. $\frac{d}{dx}\log_2(3x+1) = \frac{1}{4\ln a}\frac{du}{dx}$

$\frac{d}{dx}\log_2(u) \qquad u = 3x+1$

$a = 2 \qquad \frac{du}{dx} = 3$

$f'(x) = \frac{3}{(3x+1)\ln 2}$

$3x+1 > 0$

$x > -1/3$

42. $\frac{d}{dx}\log_{10}(\sqrt{x+1}) = \frac{1}{u\ln a}\frac{du}{dx}$

$\frac{d}{dx}\log_{10}(u) \qquad u = \sqrt{x+1}$

$a = 10 \qquad \frac{du}{dx} = \frac{1}{2\sqrt{x+1}}$

$f'(x) = \frac{1}{2(\sqrt{x+1})^2\ln 10}$

$= \frac{1}{2(x+1)\ln 10}$

$2x+2 > 0$

$x > -1$

43. $y = (\sin x)^x$

$\ln y = \ln(\sin x)^x$

$\ln y = x\ln(\sin x)$

$\frac{d}{dx}\ln y = \frac{d}{dx}\left[x\ln(\sin x)\right]$

$\frac{1}{y}\frac{dy}{dx} = (x)\left(\frac{1}{\sin x}\right)(\cos x) + \ln(\sin x)(1)$

$\frac{dy}{dx} = y\left[x\cot x + \ln(\sin x)\right]$

$\frac{dy}{dx} = (\sin x)^x[x\cot x + \ln(\sin x)]$

44. $y = x^{\tan x}$

$\ln y = \ln(x^{\tan x})$

$\ln y = (\tan x)(\ln x)$

$\frac{d}{dx}\ln y = \frac{d}{dx}\left[(\tan x)(\ln x)\right]$

$\frac{1}{y}\frac{dy}{dx} = (\tan x)\left(\frac{1}{x}\right) + (\ln x)(\sec^2 x)$

$\frac{dy}{dx} = y\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right]$

$\frac{dy}{dx} = x^{\tan x}\left[\frac{\tan x}{x} + (\ln x)(\sec^2 x)\right]$

45. $y=\sqrt[5]{\frac{(x-3)^4(x^2+1)}{(2x+5)^3}}=\left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$

$$\ln y=\ln\left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}$$

$$\ln y=\frac{1}{5}\ln\frac{(x-3)^4(x^2+1)}{(2x+5)^3}$$

$$\ln y=\frac{1}{5}[4\ln(x-3)+\ln(x^2+1)-3\ln(2x+5)]$$

$$\frac{d}{dx}(\ln y)=\frac{4}{5}\frac{d}{dx}\ln(x-3)+\frac{1}{5}\frac{d}{dx}\ln(x^2+1)-\frac{3}{5}\frac{d}{dx}\ln(2x+5)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{4}{5}\frac{1}{x-3}+\frac{1}{5}\frac{1}{x^2+1}(2x)-\frac{3}{5}\frac{1}{2x+5}(2)$$

$$\frac{dy}{dx}=y\left(\frac{4}{5(x-3)}+\frac{2x}{5(x^2+1)}-\frac{6}{5(2x+5)}\right)$$

$$\frac{dy}{dx}=\left(\frac{(x-3)^4(x^2+1)}{(2x+5)^3}\right)^{1/5}\cdot\left(\frac{4}{5(x-3)}+\frac{2x}{5(x^2+1)}-\frac{6}{5(2x+5)}\right)$$

46. $y=\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}=\frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}}$

$$\ln y=\ln\frac{x(x^2+1)^{1/2}}{(x+1)^{2/3}}$$

$$\ln y=\ln x+\frac{1}{2}\ln(x^2+1)-\frac{2}{3}\ln(x+1)$$

$$\frac{d}{dx}\ln y=\frac{d}{dx}\ln x+\frac{1}{2}\frac{d}{dx}\ln(x^2+1)-\frac{2}{3}\frac{d}{dx}\ln(x+1)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{x}+\frac{1}{2}\frac{1}{x^2+1}(2x)-\frac{2}{3}\frac{1}{x+1}(1)$$

$$\frac{dy}{dx}=y\left(\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right)$$

$$\frac{dy}{dx}=\frac{x\sqrt{x^2+1}}{(x+1)^{2/3}}\left(\frac{1}{x}+\frac{x}{x^2+1}-\frac{2}{3(x+1)}\right)$$

47. $y=x^{\ln x}$

$$\ln y=\ln x^{\ln x}$$

$$\ln y=\ln x\,\ln x$$

$$\frac{dy}{dx}(\ln y)=\frac{d}{dx}(\ln x\,\ln x)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{2\ln x}{x}$$

$$\frac{dy}{dx}=\frac{2y\ln x}{x}=\frac{2x^{\ln x}\ln x}{x}$$

48. $y=x^{(1/\ln x)}$

$$\ln y=\ln x^{(1/\ln x)}$$

$$\ln y=\frac{\ln x}{\ln x}=1$$

$$\frac{dy}{dx}(\ln y)=\frac{d}{dx}(1)$$

$$\frac{1}{y}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=0,\ x>0$$

49. The line passes through (a, e^a) for some value of a and has slope $m=e^a$. Since the line also passes through the origin, the slope is also given by e, $e\ln x<x$ or $\ln x^e<x$. and we have $e^a=\frac{e^a}{a}$, so $a=1$. Hence, the slope is e and the equation is $y=ex$.

50. For $y=xe^x$, we have $y'=(x)(e^x)+(e^x)(1)=(x+1)e^x$, so the normal line through the point (a, ae^a) has slope $m=-\frac{1}{(a+1)e^a}$ and its equation is $y=-\frac{1}{(a+1)e^a}(x-a)+ae^a$. The desired normal line includes the point (0, 0), so we have:

$$0=-\frac{1}{(a+1)e^a}(0-a)+ae^a$$

$$0=\frac{a}{(a+1)e^a}+ae^a$$

$$0=a\left(\frac{1}{(a+1)e^a}+e^a\right)$$

$$a=0\text{ or }\frac{1}{(a+1)e^a}+e^a=0$$

The equation $\frac{1}{(a+1)e^a}+e^a=0$ has no solution, so we need to use $a=0$. The equation of the normal line is $y=-\frac{1}{(0+1)e^0}(x-0)+0e^0$, or $y=-x$.

51. (a) $P(0)=\frac{300}{1+2^{4-0}}\approx 18$

(b) for all positive $x\neq e$.

$$=300\frac{d}{dt}(1+2^{4-t})^{-1}$$

$$=300\left(\frac{16\ln(2)2^t}{(2^t+16)^2}\right)$$

$$P'(4)=\frac{4800\ln(2)2^4}{(2^4+16)^2}=52$$

(c) After 4 days, the rumor will spread to 52 $\frac{\text{students}}{\text{day}}$.

52. (a) $P(0)=\dfrac{200}{1+e^{5-0}}=1$

(b) $\dfrac{d}{dt}200((1+e^{5-t})^{-1})$

$=200(-1)(1+e^{5-t})^{-2}\dfrac{d}{dt}(1+e^{5-t})$

$=200(-1)(1+e^{5-t})^{-2}(e^{5-t})(-1)$

$=\dfrac{200e^{5-t}}{(1+e^{5-t})^2}$

$P'(4)=\dfrac{200e^{5-4}}{(1+e^{5-4})^2}=39$

(c) $P'(5)=\dfrac{200e^{5-5}}{(1+e^{5-5})^2}=50$

53. $\dfrac{dA}{dt}=20\dfrac{d}{dt}\left(\dfrac{1}{2}\right)^{t/140}$

$=20\dfrac{d}{dt}2^{-t/140}$

$=20(2^{-t/140})(\ln 2)\dfrac{d}{dt}\left(-\dfrac{t}{140}\right)$

$=20(2^{-t/140})(\ln 2)\left(-\dfrac{1}{140}\right)$

$=-\dfrac{(2^{-t/140})(\ln 2)}{7}$

At $t=2$ days, we have $\dfrac{dA}{dt}=-\dfrac{(2^{-1/70})(\ln 2)}{7}\approx -0.098$ grams/day.

This means that the rate of *decay* is the positive rate of approximately 0.098 grams/day.

54. (a) $\dfrac{d}{dx}\ln(kx)=\dfrac{1}{kx}\dfrac{d}{dx}kx=\dfrac{k}{kx}=\dfrac{1}{x}$

(b) $\dfrac{d}{dx}\ln(kx)=\dfrac{d}{dx}(\ln k+\ln x)$

$=0+\dfrac{d}{dx}\ln x=\dfrac{1}{x}$

55. (a) Since $f'(x)=2^x\ln 2, f'(0)=2^0\ln 2=\ln 2$.

(b) $f'(0)=\lim_{h\to 0}\dfrac{f(h)-f(0)}{h}=\lim_{h\to 0}\dfrac{2^h-2^0}{h}$

$=\lim_{h\to 0}\dfrac{2^h-1}{h}$

(c) Since quantities in parts (a) and (b) are equal,

$\lim_{h\to 0}\dfrac{2^h-1}{h}=\ln 2.$

(d) By following the same procedure as above using $g(x)=7^x$, we may see that $\lim_{h\to 0}\dfrac{7^h-1}{h}=\ln 7.$

56. Recall that a point (a, b) is on the graph of $y=e^x$ if and only if the point (b, a) is on the graph of $y=\ln x$. Since there are points (x, e^x) on the graph of $y=e^x$ with arbitrarily large x-coordinates, there will be points $(x, \ln x)$ on the graph of $y=\ln x$ with arbitrarily large y-coordinates.

57. False. It is $\ln(2)2^x$.

58. False. It is $2e^{2x}$.

59. B. $P(0)=\dfrac{150}{1+e^{4-t}}=3$

60. D. $x+3>0$

$x>-3$

61. A. $y=\log_{10}(2x-3)$

$\dfrac{d}{dx}\log_a u=\dfrac{1}{u\ln a}\dfrac{du}{dx}$

$a=10$

$u=2x-3$

$\dfrac{du}{dx}=2$

$y'=\dfrac{2}{(2x-3)\ln 10}$

62. E. $y=2^{1-x}$

$\dfrac{d}{dx}a^u=a^u\ln a\dfrac{du}{dx}$

$a=2$

$u=1-x$

$\dfrac{du}{dx}=-1$

$y'=2^{1-x}\ln(2)(-1)$

$y'(2)=-2^{1-2}\ln(2)$

$y'(2)=-\dfrac{\ln(2)}{2}$

63. (a) The graph y_4 is a horizontal line at $y=a$.

(b) The graph of y_3 is always a horizontal line.

a	2	3	4	5
y_3	0.693147	1.098613	1.386295	1.609439
$\ln a$	0.693147	1.098612	1.386294	1.609438

We conclude that the graph of y_3 is a horizontal line at $y=\ln a$.

(c) $\dfrac{d}{dx}a^x=a^x$ if and only if $y_3=\dfrac{y_2}{y_1}=1$.

So if $y_3=\ln a$, then $\dfrac{d}{dx}a^x$ with equal a^x if and only if $\ln a=1$, or $a=e$.

(d) $y_2=\dfrac{d}{dx}a^x=a^x\ln a$. This will equal $y_1=a^x$ if and only if $\ln a=1$, or $a=e$.

64. $\dfrac{d}{dx}\left(-\dfrac{1}{2}x^2+k\right)=-x$ and $\dfrac{d}{dx}(\ln x+c)=\dfrac{1}{x}$.

Therefore, at any given value of x, these two curves will have perpendicular tangent lines.

65. **(a)** Since the line passes through the origin and has slope $\frac{1}{e}$, its equation is $y = \frac{x}{e}$.

(b) The graph of $y = \ln x$ lies below the graph of the line $y = \frac{x}{e}$ for all positive $x \neq e$. Therefore, $\ln x < \frac{x}{e}$ for all positive $x \neq e$.

(c) Multiplying by e, $e \ln x < x$ or $\ln x^e < x$.

(d) Exponentiating both sides of $\ln x^e < x$, we have $e^{\ln x^e} < e^x$, or $x^e < e^x$ for all positive $x \neq e$.

(e) Let $x = \pi$ to see that $\pi^e < e^\pi$. Therefore, e^π is bigger.

Quick Quiz Sections 3.7–3.9

1. E. $y = \frac{9}{2x} - \frac{x^2}{2}$

$dy = -\frac{9}{2x^2} - x$

$dy = -\frac{9}{2(1)^2} - 1 = -\frac{11}{2}$

2. A. $dy = \frac{d}{dx}(\cos^3(3x-2))$

$dy = -9\cos^2(3x-2)\sin(3x-2)$

3. C. $dy = \frac{d}{dx}(\sin^{-1}(2x))$

$dy = \frac{2}{\sqrt{1-4x^2}}$

4. **(a)** Differentiate implicitly:

$\frac{d}{dx}(xy^2 - x^3y) = \frac{d}{dx}(6)$

$1 \cdot y^2 + x \cdot 2y\frac{dy}{dx} - \left(3x^2y + x^3\frac{dy}{dx}\right) = 0$

$2xy\frac{dy}{dx} - x^3\frac{dy}{dx} = 3x^2y - y^2$

$\frac{dy}{dx} = \frac{3x^2y - y^2}{2xy - x^3}$

(b) If $x = 1$, then $y^2 - y = 6$, so $y = -2$ or $y = 3$.

at $(1, -2)$, $\frac{dy}{dx} = \frac{3(1)^2(-2) - (-2)^2}{2(1)(-2) - (1)^3} = 2$.

The tangent line is $y + 2 = 2(x - 1)$.

At $(1, 3)$, $\frac{dy}{dx} = \frac{3(1)^2(3) - 3^2}{2(1)(3) - 1^3} = 0$.

The tangent line is $y = 3$.

(c) The tangent line is vertical where $2xy - x^3 = 0$, which implies $x = 0$ or $y = \frac{x^2}{2}$. There is no point on the curve where $x = 0$. If $y = \frac{x^2}{2}$, then $x\left(\frac{x^2}{2}\right) - x^3\left(\frac{x^2}{2}\right) = 6$.

Then the only solution to this equation is $x = \sqrt[5]{-24}$.

Chapter 3 Review Exercises
(pp. 181–184)

1. $\frac{dy}{dx} = \frac{d}{dx}\left(x^5 - \frac{1}{8}x^2 + \frac{1}{4}x\right) = 5x^4 - \frac{1}{4}x + \frac{1}{4}$

2. $\frac{dy}{dx} = \frac{d}{dx}(3 - 7x^3 + 3x^7) = -21x^2 + 21x^6$

3. $\frac{dy}{dx} = \frac{d}{dx}(2\sin x\cos x)$

$= 2(\sin x)\frac{d}{dx}(\cos x) + 2(\cos x)\frac{d}{dx}(\sin x)$

$= -2\sin^2 x + 2\cos^2 x$

Alternate solution:

$\frac{dy}{dx} = \frac{d}{dx}(2\sin x\cos x) = \frac{d}{dx}\sin 2x = (\cos 2x)(2)$

$= 2\cos 2x$

4. $\frac{dy}{dx} = \frac{d}{dx}\frac{2x+1}{2x-1} = \frac{(2x-1)(2) - (2x-1)(2)}{(2x-1)^2} = -\frac{4}{(2x-1)^2}$

5. $\frac{ds}{dt} = \frac{d}{dt}\cos(1-2t) = -\sin(1-2t)(-2) = 2\sin(1-2t)$

6. $\frac{ds}{dt} = \frac{d}{dt}\cot\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\frac{d}{dt}\left(\frac{2}{t}\right) = -\csc^2\left(\frac{2}{t}\right)\left(-\frac{2}{t^2}\right)$

$= \frac{2}{t^2}\csc^2\left(\frac{2}{t}\right)$

7. $\frac{dy}{dx} = \frac{d}{dx}\left(\sqrt{x} + 1 + \frac{1}{\sqrt{x}}\right) = \frac{d}{dx}(x^{1/2} + 1 + x^{-1/2})$

$= \frac{1}{2}x^{-1/2} - \frac{1}{2}x^{-3/2} = \frac{1}{2\sqrt{x}} - \frac{1}{2x^{3/2}}$

8. $\frac{dy}{dx} = \frac{d}{dx}(x\sqrt{2x+1}) = (x)\left(\frac{1}{2\sqrt{2x+1}}\right)(2) + (\sqrt{2x+1})(1)$

$= \frac{x + (2x+1)}{\sqrt{2x+1}} = \frac{3x+1}{\sqrt{2x+1}}$

9. $\frac{dr}{d\theta} = \frac{d}{d\theta}\sec(1+3\theta) = \sec(1+3\theta)\tan(1+3\theta)(3)$

$= 3\sec(1+3\theta)\tan(1+3\theta)$

10. $\frac{dr}{d\theta} = \frac{d}{d\theta}\tan^2(3-\theta^2)$

$= 2\tan(3-\theta^2)\frac{d}{d\theta}\tan(3-\theta^2)$

$= 2\tan(3-\theta^2)\sec^2(3-\theta^2)(-2\theta)$

$= -4\theta\tan(3-\theta^2)\sec^2(3-\theta^2)$

11. $\frac{dy}{dx} = \frac{d}{dx}(x^2\csc 5x)$

$= (x^2)(-\csc 5x\cot 5x)(5) + (\csc 5x)(2x)$

$= -5x^2\csc 5x\cot 5x + 2x\csc 5x$

12. $\frac{dy}{dx}=\frac{d}{dx}\ln\sqrt{x}=\frac{1}{\sqrt{x}}\frac{d}{dx}\sqrt{x}=\frac{1}{\sqrt{x}}\cdot\frac{1}{2\sqrt{x}}=\frac{1}{2x},\ x>0$

13. $\frac{dy}{dx}=\frac{d}{dx}\ln(1+e^x)=\frac{1}{1+e^x}\frac{d}{dx}(1+e^x)=\frac{e^x}{1+e^x}$

14. $\frac{dy}{dx}=\frac{d}{dx}(xe^{-x})=(x)(e^{-x})(-1)+(e^{-x})(1)=-xe^{-x}+e^{-x}$

15. $\frac{dy}{dx}=\frac{d}{dx}(e^{1+\ln x})=\frac{d}{dx}(e^1e^{\ln x})=\frac{d}{dx}(ex)=e$

16. $\frac{dy}{dx}=\frac{d}{dx}\ln(\sin x)=\frac{1}{\sin x}\frac{d}{dx}(\sin x)=\frac{\cos x}{\sin x}=\cot x$, for values of x in the intervals $(k\pi,\ (k+1)\pi)$, where k is even.

17. $\frac{dr}{dx}=\frac{d}{dx}\ln(\cos^{-1}x)=\frac{1}{\cos^{-1}x}\frac{d}{dx}\cos^{-1}x$

$$=\frac{1}{\cos^{-1}x}\left(-\frac{1}{\sqrt{1-x^2}}\right)=-\frac{1}{\cos^{-1}x\sqrt{1-x^2}}$$

18. $\frac{dr}{d\theta}=\frac{d}{d\theta}\log_2(\theta^2)=\frac{1}{\theta^2\ln 2}\frac{d}{d\theta}(\theta^2)=\frac{2\theta}{\theta^2\ln 2}=\frac{2}{\theta\ln 2}$

19. $\frac{ds}{dt}=\frac{d}{dt}\log_5(t-7)=\frac{1}{(t-7)\ln 5}\frac{d}{dt}(t-7)=\frac{1}{(t-7)\ln 5}$, $t>7$

20. $\frac{ds}{dt}=\frac{d}{dt}(8^{-t})=8^{-t}(\ln 8)\frac{d}{dt}(-t)=-8^{-t}\ln 8$

21. Use logarithmic differentiation.

$$y=x^{\ln x}$$
$$\ln y=\ln(x^{\ln x})$$
$$\ln y=(\ln x)(\ln x)$$
$$\frac{d}{dx}\ln y=\frac{d}{dx}(\ln x)^2$$
$$\frac{1}{y}\frac{dy}{dx}=2\ln x\frac{d}{dx}\ln x$$
$$\frac{dy}{dx}=\frac{2y\ln x}{x}$$
$$\frac{dy}{dx}=\frac{2x^{\ln x}\ln x}{x}$$

22. $\frac{dy}{dx}=\frac{d}{dx}\frac{(2x)2^x}{\sqrt{x^2+1}}$

$$=\frac{\sqrt{x^2+1}\frac{d}{dx}\left[(2x)2^x\right]-(2x)(2^x)\frac{d}{dx}\sqrt{x^2+1}}{x^2+1}$$

$$=\frac{\sqrt{x^2+1}\left[(2x)(2^x)(\ln 2)+(2^x)(2)\right]-(2x)(2^x)\frac{1}{2\sqrt{x^2+1}}(2x)}{x^2+1}$$

$$=\frac{(x^2+1)(2^x)(2x\ln 2+2)-2x^2(2^x)}{(x^2+1)^{3/2}}$$

$$=\frac{(2\cdot 2^x)[(x^2+1)(x\ln 2+1)-x^2]}{(x^2+1)^{3/2}}$$

$$=\frac{(2\cdot 2^x)(x^3\ln 2+x^2+x\ln 2+1-x^2)}{(x^2+1)^{3/2}}$$

$$=\frac{(2\cdot 2^x)(x^3\ln 2+x\ln 2+1)}{(x^2+1)^{3/2}}$$

Alternate solution, using logarithmic differentiation:

$$y=\frac{(2x)2^x}{\sqrt{x^2+1}}$$
$$\ln y=(2x)+\ln(2^x)-\ln\sqrt{x^2+1}$$
$$\ln y=\ln 2+\ln x+x\ln 2-\frac{1}{2}\ln(x^2+1)$$
$$\frac{d}{dx}\ln y=\frac{d}{dx}[\ln 2+\ln x+\ln 2-\frac{1}{2}\ln(x^2+1)]$$
$$\frac{1}{y}\frac{dy}{dx}=0+\frac{1}{x}+\ln 2-\frac{1}{2}\frac{1}{x^2+1}(2x)$$
$$\frac{dy}{dx}=y\left(\frac{1}{x}+\ln 2-\frac{x}{x^2+1}\right)$$
$$\frac{dy}{dx}=\frac{(2x)2^x}{\sqrt{x^2+1}}\left(\frac{1}{x}+\ln 2-\frac{x}{x^2+1}\right)$$

23. $\frac{dy}{dx}=\frac{d}{dx}e^{\tan^{-1}x}=e^{\tan^{-1}x}\frac{d}{dx}\tan^{-1}x=\frac{e^{\tan^{-1}x}}{1+x^2}$

24. $\frac{dy}{du}=\frac{d}{dx}\sin^{-1}\sqrt{1-u^2}$

$$=\frac{1}{\sqrt{1-(\sqrt{1-u^2})^2}}\frac{d}{du}\sqrt{1-u^2}$$
$$=\frac{1}{\sqrt{u^2}}\frac{1}{2\sqrt{1-u^2}}(-2u)=\frac{u}{|u|\sqrt{1-u^2}}$$

25. $\frac{dy}{dt}=\frac{d}{dt}\left(t\sec^{-1}t-\frac{1}{2}\ln t\right)$

$$=(t)\left(\frac{1}{|t|\sqrt{t^2-1}}\right)+(\sec^{-1}t)(1)-\frac{1}{2t}$$
$$=\frac{t}{|t|\sqrt{t^2-1}}+\sec^{-1}t-\frac{1}{2t}$$

26. $\frac{dy}{dt}=\frac{d}{dt}[(1+t^2)\cot^{-1}2t]$

$$=(1+t^2)\left(-\frac{1}{1+(2t)^2}\right)(2)+(\cot^{-1}2t)(2t)$$
$$=-\frac{2+2t^2}{1+4t^2}+2t\cot^{-1}2t$$

27. $\frac{dy}{dz} = \frac{d}{dz}(z\cos^{-1} z - \sqrt{1-z^2})$

$= (z)\left(-\frac{1}{\sqrt{1-z^2}}\right) + (\cos^{-1} z)(1) - \frac{1}{2\sqrt{1-z^2}}(-2z)$

$= -\frac{z}{\sqrt{1-z^2}} + \cos^{-1} z + \frac{z}{\sqrt{1-z^2}}$

$= \cos^{-1} z$

28. $\frac{dy}{dx} = \frac{d}{dx}(2\sqrt{x-1}\csc^{-1}\sqrt{x})$

$= (2\sqrt{x-1})\left(-\frac{1}{|\sqrt{x}|\sqrt{(\sqrt{x})^2-1}}\right)\left(\frac{1}{2\sqrt{x}}\right)$

$+ (2\csc^{-1}\sqrt{x})\left(\frac{1}{2\sqrt{x-1}}\right)$

$= -\frac{\sqrt{x-1}}{(\sqrt{x})^2\sqrt{x-1}} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$

$= -\frac{1}{x} + \frac{\csc^{-1}\sqrt{x}}{\sqrt{x-1}}$

29. $\frac{dy}{dx} = \frac{d}{dx}\csc^{-1}(\sec x)$

$= \left(-\frac{1}{|\sec x|\sqrt{\sec^2 x - 1}}\right)\frac{d}{dx}(\sec x)$

$= -\frac{1}{|\sec x|\sqrt{\tan^2 x - 1}}\sec x \tan x$

$= -\frac{\sec x \tan x}{|\sec x \tan x|}$

$= -\frac{\frac{1}{\cos x}\frac{\sin x}{\cos x}}{\left|\frac{1}{\cos x}\frac{\sin x}{\cos x}\right|} = -\frac{\sin x}{|\sin x|}$

$= \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \frac{\pi}{2} \\ 1, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$

Alternate method:

On the domain $0 \le x \le 2\pi, x \ne \frac{\pi}{2}, x \ne \frac{3\pi}{2}$, we may rewrite the function as follows:

$y = \csc^{-1}(\sec x)$

$= \frac{\pi}{2} - \sec^{-1}(\sec x)$

$= \frac{\pi}{2} - \cos^{-1}(\cos x)$

$= \begin{cases} \frac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \frac{\pi}{2} \\ \frac{\pi}{2} - (\pi - x), & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$

$= \begin{cases} \frac{\pi}{2} - x, & 0 \le x \le \pi, \quad x \ne \frac{\pi}{2} \\ \frac{\pi}{2} - (\pi + x), & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$

Therefore, $\frac{dy}{dx} = \begin{cases} -1, & 0 \le x < \pi, \quad x \ne \frac{\pi}{2} \\ 1, & \pi < x \le 2\pi, \quad x \ne \frac{3\pi}{2} \end{cases}$

Note that the derivative exists at 0 and 2π only because these are the endpoints of the given domain; the two-sided derivative of $y = \csc^{-1}(\sec x)$ does not exist at these points.

30. $\frac{dr}{d\theta} = \frac{d}{d\theta}\left(\frac{1+\sin\theta}{1-\cos\theta}\right)^2$

$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{(1-\cos\theta)(\cos\theta) - (1+\sin\theta)(\sin\theta)}{(1-\cos\theta)^2}\right)$

$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{\cos\theta - \cos^2\theta - \sin\theta - \sin^2\theta}{(1-\cos\theta)^2}\right)$

$= 2\left(\frac{1+\sin\theta}{1-\cos\theta}\right)\left(\frac{\cos\theta - \sin\theta - 1}{(1-\cos\theta)^2}\right)$

31. Since $y = \ln x^2$ is defined for all $x \ne 0$ and $\frac{dy}{dx} = \frac{1}{x^2}\frac{d}{dx}(x^2) = \frac{2x}{x^2} = \frac{2}{x}$, the function is differentiable for all $x \ne 0$.

32. Since $y = \sin x - x\cos x$ is defined for all real x and $\frac{dy}{dx} = \cos x - (x)(-\sin x) - (\cos x)(1) = x\sin x$, the function is differentiable for all real x.

33. Since $y = \sqrt{\frac{1-x}{1+x^2}}$ is defined for all $x < 1$ and

$\frac{dy}{dx} = \frac{1}{2\sqrt{\frac{1-x}{1+x^2}}}\frac{(1+x^2)(-1) - (1-x)(2x)}{(1+x^2)^2}$

$= \frac{x^2 - 2x - 1}{2\sqrt{1-x}(1+x^2)^{3/2}}$, which is defined only for $x < 1$, the function is differentiable for all $x < 1$.

34. Since $y = (2x-7)^{-1}(x+5) = \frac{x+5}{2x-7}$ is defined for all $x \ne \frac{7}{2}$ and $\frac{dy}{dx} = \frac{(2x-7)(1) - (x+5)(2)}{(2x-7)^2} = -\frac{17}{(2x-7)^2}$, the function is differentiable for all $x \ne \frac{7}{2}$.

35. Use implicit differentiation.

$$xy + 2x + 3y = 1$$

$$\frac{dy}{dx}(xy) + \frac{d}{dx}(2x) + \frac{d}{dx}(3y) = \frac{d}{dx}(1)$$

$$x\frac{d}{dx} + (y)(1) + 2 + 3\frac{dy}{dx} = 0$$

$$(x+3)\frac{dy}{dx} = -(y+2)$$

$$\frac{dy}{dx} = -\frac{y+2}{x+3}$$

36. Use implicit differentiation.

$$5x^{4/5}+10y^{6/5}=15$$
$$\frac{d}{dx}(5x^{4/5})+\frac{d}{dx}(10y^{6/5})=\frac{d}{dx}(15)$$
$$4x^{-1/5}+12y^{1/5}\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{4x^{-1/5}}{12y^{1/5}}=-\frac{1}{3(xy)^{1/5}}$$

37. Using implicit differentiation.

$$\sqrt{xy}=1$$
$$\frac{d}{dx}\sqrt{xy}=\frac{d}{dx}(1)$$
$$\frac{1}{2\sqrt{xy}}\left[x\frac{dy}{dx}+(y)(1)\right]=0$$
$$x\frac{dy}{dx}+y=0$$
$$\frac{dy}{dx}=-\frac{y}{x}$$

Alternate method:

38. Use implicit differentiation.

$$y^2=\frac{x}{x+1}$$
$$\frac{d}{dx}y^2=\frac{d}{dx}\frac{x}{x+1}$$
$$2y\frac{dy}{dx}=\frac{(x+1)(1)-(x)(1)}{(x+1)^2}$$
$$\frac{dy}{dx}=\frac{1}{2y(x+1)^2}$$

39.

$$x^3+y^3=1$$
$$\frac{d}{dx}(x^3)+\frac{d}{dx}(y^3)=\frac{d}{dx}(1)$$
$$3x^2+3y^2y'=0$$
$$y'=-\frac{x^2}{y^2}$$
$$y''=\frac{d}{dx}\left(-\frac{x^2}{y^2}\right)$$
$$=-\frac{(y^2)(2x)-(x^2)(2y)(y')}{y^4}$$
$$=-\frac{(y^2)(2x)-(x^2)(2y)\left(-\frac{x^2}{y^2}\right)}{y^4}$$
$$=-\frac{2xy^3+2x^4}{y^5}$$
$$=-\frac{2x(x^3+y^3)}{y^5}$$
$$=-\frac{2x}{y^5}$$

since $x^3+y^3=1$

40.

$$y^2=1-\frac{2}{x}$$
$$\frac{d}{dx}(y^2)=\frac{d}{dx}(1)-\frac{d}{dx}\left(\frac{2}{x}\right)$$
$$2yy'=\frac{2}{x^2}$$
$$y'=\frac{2}{x^{2(2y)}}=\frac{1}{x^2y}$$
$$y''=\frac{d}{dx}\left(\frac{1}{x^2y}\right)$$
$$=-\frac{1}{(x^2y)^2}\frac{d}{dx}(x^2y)$$
$$=-\frac{1}{(x^2y)^2}[(x^2)(y')+(y)(2x)]$$
$$=-\frac{1}{(x^2y)^2}\left[(x^2)\left(\frac{1}{x^2y}\right)+2xy\right]$$
$$=-\frac{1}{x^4y^2}\left(\frac{1}{y}+2xy\right)$$
$$=-\frac{1+2xy^2}{x^4y^3}$$

41.

$$y^3+y=2\cos x$$
$$\frac{d}{dx}(y^3)+\frac{d}{dx}(y)=\frac{d}{dx}(2\cos x)$$
$$3y^2y'+y'=-2\sin x$$
$$(3y^2+1)y'=-2\sin x$$
$$y'=-\frac{2\sin x}{3y^2+1}$$
$$y''=\frac{d}{dx}\left(-\frac{2\sin x}{3y^2+1}\right)$$
$$=-\frac{(3y^2+1)(2\cos x)-(2\sin x)(6yy')}{(3y^2+1)^2}$$
$$=-\frac{(3y^2+1)(2\cos x)-(12y\sin x)\left(-\frac{2\sin x}{3y^2+1}\right)}{(3y^2+1)^2}$$
$$=-2\frac{(3y^2+1)^2\cos x+12y\sin^2 x}{(3y^2+1)^3}$$

42.
$$x^{1/3}+y^{1/3}=4$$
$$\frac{d}{dx}(x^{1/3})+\frac{d}{dx}(y^{1/3})=\frac{d}{dx}(4)$$
$$\frac{1}{3}x^{-2/3}+\frac{1}{3}y^{-2/3}y'=0$$
$$y'=-\frac{x^{-2/3}}{y^{-2/3}}=-\left(\frac{y}{x}\right)^{2/3}$$
$$y''=\frac{d}{dx}\left[-\left(\frac{y}{x}\right)^{2/3}\right]$$
$$=-\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{xy'-(y)(1)}{x^2}\right)$$
$$=-\frac{2}{3}\left(\frac{y}{x}\right)^{-1/3}\left(\frac{(x)\left[-\left(\frac{y}{x}\right)^{2/3}\right]-y}{x^2}\right)$$
$$=-\frac{2}{3}x^{1/3}y^{-1/3}(-x^{-5/3}y^{2/3}-x^{-2}y)$$
$$=\frac{2}{3}x^{-4/3}y^{1/3}+\frac{2}{3}x^{-5/3}y^{2/3}$$

43. $y'=2x^3-3x-1,$
$y''=6x^2-3,$
$y'''=12x,$
$y^{(4)}=12,$ and the rest are all zero.

44. $y'=\frac{x^4}{24},$
$y''=\frac{x^3}{6},$
$6y'''=\frac{x^2}{2},$
$y^{(4)}=x,$
$y^{(5)}=1,$ and the rest are all zero.

45. $\frac{dy}{dx}=\frac{d}{dx}\sqrt{x^2-2x}=\frac{1}{2\sqrt{x^2-2x}}(2x-2)=\frac{x-1}{\sqrt{x^2-2x}}$

At $x=3$, we have $y=\sqrt{3^2-2(3)}=\sqrt{3}$

and $\frac{dy}{dx}=\frac{3-1}{\sqrt{3^2-2(3)}}=\frac{2}{\sqrt{3}}.$

(a) Tangent: $y=\frac{2}{\sqrt{3}}(x-3)+\sqrt{3}$ or $y=\frac{2}{\sqrt{3}}x-\sqrt{3}$

(b) Normal: $y=-\frac{\sqrt{3}}{2}(x-3)+\sqrt{3}$ or $y=-\frac{\sqrt{3}}{2}x+\frac{5\sqrt{3}}{2}$

46. $\frac{dy}{dx}=\frac{d}{dx}(4+\cot x-2\csc x)$
$=-\csc^2 x+2\csc x\cot x$

At $x=\frac{\pi}{2}$, we have

$y=4+\cot\frac{\pi}{2}-2\csc\frac{\pi}{2}=4+0-2=2$ and

$\frac{dy}{dx}=-\csc^2\frac{\pi}{2}+2\csc\frac{\pi}{2}\cot\frac{\pi}{2}=-1+2(1)(0)=-1.$

(a) Tangent: $y=-1\left(x-\frac{\pi}{2}\right)+2$ or $y=-x+\frac{\pi}{2}+2$

(b) Normal: $y=-1\left(x-\frac{\pi}{2}\right)+2$ or $y=x-\frac{\pi}{2}+2$

47. Use implicit differentiation.
$$x^2+2y^2=9$$
$$\frac{d}{dx}(x^2)+\frac{d}{dx}(2y^2)=\frac{d}{dx}(9)$$
$$2x+4y\frac{dy}{dx}=0$$
$$\frac{dy}{dx}=-\frac{2x}{4y}=-\frac{x}{2y}$$

Slope at (1, 2): $-\frac{1}{2(2)}=-\frac{1}{4}$

(a) Tangent: $y=-\frac{1}{4}(x-1)+2$ or $y=-\frac{1}{4}x+\frac{9}{4}$

(b) Normal: $y=4(x-1)+2$ or $y=4x-2$

48. Use implicit differentiation.
$$x+\sqrt{xy}=6$$
$$\frac{d}{dx}(x)+\frac{d}{dx}(\sqrt{xy})=\frac{d}{dx}(6)$$
$$1+\frac{1}{2\sqrt{xy}}\left[(x)\left(\frac{dy}{dx}\right)+(y)(1)\right]=0$$
$$\frac{x}{2\sqrt{xy}}\frac{dy}{dx}=-1-\frac{y}{2\sqrt{xy}}$$
$$\frac{dy}{dx}=\frac{2\sqrt{xy}}{x}\left(-1-\frac{y}{2\sqrt{xy}}\right)$$
$$=-2\sqrt{\frac{y}{x}}-\frac{y}{x}$$

Slope at (4, 1): $-2\sqrt{\frac{1}{4}}-\frac{1}{4}=-\frac{2}{2}-\frac{1}{4}=-\frac{5}{4}$

(a) Tangent: $y=-\frac{5}{4}(x-4)+1$ or $y=-\frac{5}{4}x+6$

(b) Normal: $y=-\frac{4}{5}(x-4)+1$ or $y=-\frac{4}{5}x-\frac{11}{5}$

49. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-2\sin t}{2\cos t} = -\tan t$

At $t = \frac{3\pi}{4}$, we have $x = 2\sin\frac{3\pi}{4} = \sqrt{2}$,

$y = 2\cos\frac{3\pi}{4} = -\sqrt{2}$, and $\frac{dy}{dx} = -\tan\frac{3\pi}{4} = 1$.

The equation of the tangent line is

$y = 1(x - \sqrt{2}) + (-\sqrt{2})$, or $y = x - 2\sqrt{2}$.

50. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{4\cos t}{-3\sin t} = -\frac{4}{3}\cot t$

At $t = \frac{3\pi}{4}$, we have $x = 3\cos\frac{3\pi}{4} = -\frac{3\sqrt{2}}{2}$,

$y = 4\sin\frac{3\pi}{4} = 2\sqrt{2}$, and $\frac{dy}{dx} = -\frac{4}{3}\cot\frac{3\pi}{4} = \frac{4}{3}$.

The equation of the tangent line is

$y = \frac{4}{3}\left(x + \frac{3\sqrt{2}}{2}\right) + 2\sqrt{2}$, or $y = \frac{4}{3}x + 4\sqrt{2}$.

51. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{5\sec^2 t}{3\sec t\tan t} = \frac{5\sec t}{3\tan t} = \frac{5}{3\sin t}$

At $t = \frac{\pi}{6}$, we have $x = 3\sec\frac{\pi}{6} = 2\sqrt{3}$,

$y = 5\tan\frac{\pi}{6} = \frac{5\sqrt{3}}{3}$, and $\frac{dy}{dx} = \frac{5}{3\sin\left(\frac{\pi}{6}\right)} = \frac{10}{3}$.

The equation of the tangent line is

$y = \frac{10}{3}(x - 2\sqrt{3}) + \frac{5\sqrt{3}}{3}$, or $y = \frac{10}{3}x - 5\sqrt{3}$.

52. $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{1+\cos t}{-\sin t}$

At $t = -\frac{\pi}{4}$, we have $x = \cos\left(-\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}$,

$y = -\frac{\pi}{4} + \sin\left(-\frac{\pi}{4}\right) = -\frac{\pi}{4} - \frac{\sqrt{2}}{2}$, and

$\frac{dy}{dx} = \frac{1+\cos\left(-\frac{\pi}{4}\right)}{-\sin\left(-\frac{\pi}{4}\right)} = \frac{1+\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = \sqrt{2} + 1$.

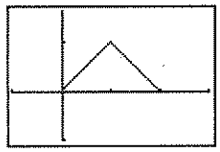
The equation of the tangent line is

$y = (\sqrt{2}+1)\left(x - \frac{\sqrt{2}}{2}\right) - \frac{\pi}{4} - \frac{\sqrt{2}}{2}$, or

$y = (1+\sqrt{2})x - \sqrt{2} - 1 - \frac{\pi}{4}$.

This is approximately $y = 2.414x - 3.200$.

53. **(a)**



[-1, 3] by [-1, 5/3]

(b) Yes, because both of the one-sided limits as $x \to 1$ are equal to $f(1) = 1$.

(c) No, because the left-hand derivative at $x = 1$ is $+1$ and the right-hand derivative at $x = 1$ is -1.

54. **(a)** The function is continuous for all values of m, because the right-hand limit as $x \to 0$ is equal to $f(0) = 0$ for any value of m.

(b) The left-hand derivative at $x = 0$ is $2\cos(2 \bullet 0) = 2$, and the right-hand derivative at $x = 0$ is m, so in order for the function to be differentiable at $x = 0$, m must be 2.

55. **(a)** For all $x \neq 0$ **(b)** At $x = 0$

(c) Nowhere

56. **(a)** For all x **(b)** Nowhere

(c) Nowhere

57. Note that $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(2x-3) = -3$ and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x-3) = -3$. Since these values agree with $f(0)$, the function is continuous at $x = 0$. On the other hand,

$f'(x) = \begin{cases} 2, & -1 \le x < 0 \\ 1, & 0 < x \le 4 \end{cases}$, so the derivative is undefined at $x = 0$.

(a) $[-1, 0) \cup (0, 4]$ **(b)** At $x = 0$

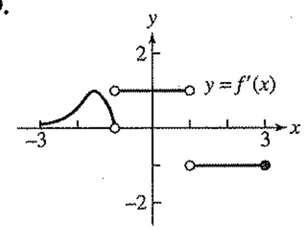
(c) Nowhere in its domain

58. Note that the function is undefined at $x = 0$.

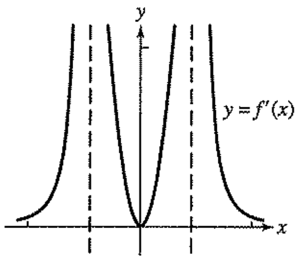
(a) $[-2, 0) \cup (0, 2]$ **(b)** Nowhere

(c) Nowhere in its domain

59.

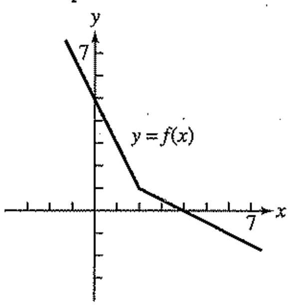


60.

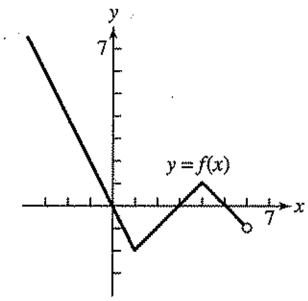


61. **(a)** iii **(b)** i
(c) ii

62. The graph passes through (0, 5) and has slope –2 for $x < 2$ and slope –0.5 for $x > 2$.



63. The graph passes through (–1, 2) and has slope –2 for $x < 1$, slope 1 for $1 < x < 4$, and slope –1 for $4 < x < 6$.



64. **i.** If $f(x)=\frac{9}{28}x^{7/3}+9$, then $f'(x)=\frac{3}{4}x^{4/3}$ and $f''(x)=x^{1/3}$, which matches the given equation.

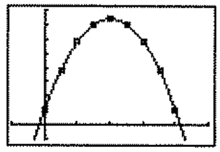
ii. If $f'(x)=\frac{9}{28}x^{7/3}-2$, then $f''(x)=\frac{3}{4}x^{4/3}$, which contradicts the given equation $f''(x)=x^{1/3}$.

iii. If $f'(x)=\frac{3}{4}x^{4/3}+6$, then $f''(x)=x^{1/3}$, which matches the given equation.

iv. If $f(x)=\frac{3}{4}x^{4/3}-4$, then $f'(x)=x^{1/3}$ and $f''(x)=\frac{1}{3}x^{-2/3}$, which contradicts the given equation $f''(x)=x^{1/3}$.

Answer is D: i and iii only could be true. Note, however that i and iii could not simultaneously be true.

65. **(a)**

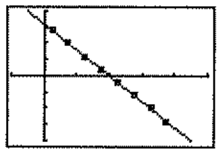


[-1, 5] by [-10,80]

(b)

t interval	avg. vel.
[0, 0.5]	$\frac{38-10}{0.5-0}=56$
[0.5, 1]	$\frac{58-38}{1-0.5}=40$
[1, 1.5]	$\frac{70-58}{1.5-1}=24$
[1.5, 2]	$\frac{74-70}{2-1.5}=8$
[2, 2.5]	$\frac{70-74}{2.5-2}=-8$
[2.5, 3]	$\frac{58-70}{3-2.5}=-24$
[3, 3.5]	$\frac{38-58}{3.5-3}=-40$
[3.5, 4]	$\frac{10-38}{4-3.5}=-56$

(c)



[-1, 5] by [-80, 80]

(d) Average velocity is a good approximation to velocity.

66. **(a)** $\frac{d}{dx}\left[\sqrt{x}f(x)\right]=\sqrt{x}f'(x)+\frac{1}{2\sqrt{x}}f(x)$

At $x=1$, the derivative is

$$\sqrt{1}f'(1)+\frac{1}{2\sqrt{1}}f(1)=1\left(\frac{1}{5}\right)+\left(\frac{1}{2}\right)(-3)=-\frac{13}{10}.$$

(b) $\frac{d}{dx}\sqrt{f(x)}=\frac{1}{2\sqrt{f(x)}}f'(x)=\frac{f'(x)}{2\sqrt{f(x)}}$

At $x=0$, the derivative is $\frac{f'(0)}{2\sqrt{f(0)}}=-\frac{2}{2\sqrt{9}}=-\frac{1}{3}.$

(c) $\frac{d}{dx}f(\sqrt{x})=f'(\sqrt{x})\frac{d}{dx}\sqrt{x}=\frac{f'(\sqrt{x})}{2\sqrt{x}}$

At $x=1$, the derivative is $\frac{f'(\sqrt{1})}{2\sqrt{1}}=\frac{f'(1)}{2}=\frac{\frac{1}{5}}{2}=\frac{1}{10}.$

(d) $\frac{d}{dx}f(1-5\tan x)=f'(1-5\tan x)(-5\sec^2 x)$

At $x=0$, the derivative is

$$f'(1-5\tan 0)(-5\sec^2 0)=f'(1)(-5)=\left(\frac{1}{5}\right)(-5)=-1.$$

66. Continued

(e) $\frac{d}{dx}\frac{f(x)}{2+\cos x}=\frac{(2+\cos x)(f'(x))-(f(x))(-\sin x)}{(2+\cos x)^2}$

At $x=0$, the derivative is

$\frac{(2+\cos 0)(f'(0))-(f(0))(-\sin 0)}{(2+\cos 0)^2}=\frac{3f'(0)}{3^2}=-\frac{2}{3}.$

(f) $\frac{d}{dx}[10\sin\left(\frac{\pi x}{2}\right)f^2(x)]$

$=10\left(\sin\frac{\pi x}{2}\right)(2f(x)f'(x))+10f^2(x)\left(\cos\frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)$

$=20f(x)f'(x)\sin\frac{\pi x}{2}+5\pi f^2(x)\cos\frac{\pi x}{2}$

At $x=1$, the derivative is

$20f(1)f'(1)\sin\frac{\pi}{2}+5\pi f^2(1)\cos\frac{\pi}{2}$

$=20(-3)\left(\frac{1}{5}\right)(1)+5\pi(-3)^2(0)$

$=-12.$

67. (a) $\frac{d}{dx}\left[3f(x)-g(x)\right]=3f'(x)-g'(x)$

At $x=-1$, the derivative is

$3f'(-1)-g'(-1)=3(2)-1=5.$

(b) $\frac{d}{dx}\left[f^2(x)g^3(x)\right]$

$=f^2(x)\cdot 3g^2(x)g'(x)+g^3(x)\cdot 2f(x)f'(x)$

$=f(x)g^2(x)\left[3f(x)g'(x)+2g(x)f'(x)\right]$

At $x=0$, the derivative is

$f(0)g^2(0)\left[3f(0)g'(0)+2g(0)f'(0)\right]$

$=(-1)(-3)^2\left[3(-1)(4)+2(-3)(-2)\right]$

$=-9\left[-12+12\right]=0.$

(c) $\frac{d}{dx}g(f(x))=g'(f(x))f'(x)$

At $x=-1$, the derivative is

$g'(f(-1))f'(-1)=g'(0)f'(-1)=(4)(2)=8.$

(d) $\frac{d}{dx}f(g(x))=f'(g(x))g'(x)$

At $x=-1$, the derivative is

$f'(g(-1))g'(-1)=f'(-1)g'(-1)=(2)(1)=2.$

(e) $\frac{d}{dx}\frac{f(x)}{g(x)+2}=\frac{(g(x)+2)f'(x)-f(x)g'(x)}{(g(x)+2)^2}$

At $x=0$, the derivative is

$\frac{(g(0)+2)f'(0)-f(0)g'(0)}{(g(0)+2)^2}=\frac{(-3+2)(-2)-(-1)(4)}{(-3+2)^2}$

$=6.$

(f) $\frac{d}{dx}g(x+f(x))=g'(x+f(x))\frac{d}{dx}(x+f(x))$

$=g'(x+f(x))(1+f'(x))$

At $x=0$, the derivative is $g'(0+f(0))[1+f'(0)]$

$=g'(0-1)[1+(-2)]=(1)(-1)=-1$

68. $\frac{dw}{ds}=\frac{dw}{dr}\frac{dr}{ds}=\frac{d}{dr}[\sin(\sqrt{r}-2)]\frac{d}{ds}\left[8\sin\left(s+\frac{\pi}{6}\right)\right]$

$=\left[\cos(\sqrt{r}-2)\frac{1}{2\sqrt{r}}\right]\left[8\cos\left(s+\frac{\pi}{6}\right)\right]$

At $s=0$, we have $r=8\sin\left(0+\frac{\pi}{6}\right)=4$ and so

$\frac{dw}{ds}=\left[\cos(\sqrt{4}-2)\frac{1}{2\sqrt{4}}\right]\left[8\cos\left(0+\frac{\pi}{6}\right)\right]$

$=\left(\frac{\cos 0}{4}\right)\left(8\cos\frac{\pi}{6}\right)=\left(\frac{1}{4}\right)\left(\frac{8\sqrt{3}}{2}\right)=\sqrt{3}$

69. Solving $\theta^2 t+\theta=1$ for t, we have

$t=\frac{1-\theta}{\theta^2}=\theta^{-2}-\theta^{-1}$, and we may write:

$$\frac{dr}{d\theta}=\frac{dr}{dt}\frac{dt}{d\theta}$$

$$\frac{d}{d\theta}(\theta^2+7)^{1/3}=\frac{dr}{dt}\frac{d}{d\theta}(\theta^{-2}-\theta^{-1})$$

$$\frac{1}{3}(\theta^2+7)^{-2/3}(2\theta)=\left(\frac{dr}{dt}\right)(-2\theta^{-3}+\theta^{-2})$$

$$\frac{dr}{dt}=\frac{2\theta(\theta^2+7)^{-2/3}}{3(-2\theta^{-3}+\theta^{-2})}=\frac{2\theta^4(\theta^2+7)^{-2/3}}{3(\theta-2)}$$

At $t=0$, we may solve $\theta^2 t+\theta=1$ to obtain $\theta=1$,

and so $\frac{dr}{dt}=\frac{2(1)^4(1^2+7)^{-2/3}}{3(1-2)}=\frac{2(8)^{-2/3}}{-3}=-\frac{1}{6}.$

70. (a) One possible answer:

$x(t)=10\cos\left(t+\frac{\pi}{4}\right),\ y(t)=1$

(b) $s(0)=10\cos\frac{\pi}{4}=5\sqrt{2}$

(c) Farthest left:

When $\cos\left(t+\frac{\pi}{4}\right)=-1$, we have $s(t)=-10.$

Farthest right:

When $\cos\left(t+\frac{\pi}{4}\right)=1$, we have $s(t)=10.$

70. Continued

(d) Since $\cos\frac{\pi}{2}=0$, the particle first reaches the origin at $t=\frac{\pi}{4}$. The velocity is given by $v(t)=-10\sin\left(t+\frac{\pi}{4}\right)$, so the velocity at $t=\frac{\pi}{4}$ is $-10\sin\frac{\pi}{2}=-10$, and the speed at $t=\frac{\pi}{4}$ is $|-10|=10$. The acceleration is given by $a(t)=-10\cos\left(t+\frac{\pi}{4}\right)$, so the acceleration at $t=\frac{\pi}{4}$ is $-10\cos\frac{\pi}{2}=0$.

71. (a) $\frac{ds}{dt}=\frac{d}{dt}(64t-16t^2)=64-32t$

$\frac{d^2s}{dt^2}=\frac{d}{dt}(64-32t)=-32$

(b) The maximum height is reached when $\frac{ds}{dt}=0$, which occurs at $t=2$ sec.

(c) When $t=0$, $\frac{ds}{dt}=64$, so the velocity is 64 ft/sec.

(d) Since $\frac{ds}{dt}=\frac{d}{dt}(64t-2.6t^2)=64-5.2t$, the maximum height is reached at $t=\frac{64}{5.2}\approx 12.3\text{sec}$. The maximum height is $s\left(\frac{64}{5.2}\right)\approx 393.8$ ft.

72. (a) Solving $160=490t^2$, it takes $\frac{4}{7}$ sec. The average velocity is $\frac{160}{4}=280\text{cm/sec}$.

(b) Since $v(t)=\frac{ds}{dt}=980t$, the velocity is $(980)\left(\frac{4}{7}\right)=560$ cm/sec. Since $a(t)=\frac{dv}{dt}=980$, the acceleration is $980\text{ cm}/\text{sec}^2$.

73. $\frac{dV}{dx}=\frac{d}{dx}\left[\pi\left(10-\frac{x}{3}\right)x^2\right]=\frac{d}{dx}\left[\pi\left(10x^2-\frac{1}{3}x^3\right)\right]$

$=\pi(20x-x^2)$

74. (a) $r(x)=\left(3-\frac{x}{40}\right)^2x=9x-\frac{3}{20}x^2+\frac{1}{1600}x^3$

(b) The marginal revenue is

$r'(x)=9-\frac{3}{10}x+\frac{3}{1600}x^2$

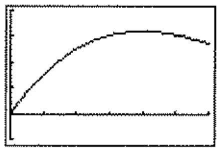
$=\frac{3}{1600}(x^2-160x+4800)$

$=\frac{3}{1600}(x-40)(x-120)$,

which is zero when $x=40$ or $x=120$. Since the bus holds only 60 people, we require $0\le x\le 60$. The marginal revenue is 0 when there are 40 people, and the corresponding fare is $p(40)=\left(3-\frac{40}{40}\right)^2=\$4.00$.

(c) One possible answer:

If the current ridership is less than 40, then the proposed plan may be good. If the current ridership is greater than or equal to 40, then the plan is not a good idea. Look at the graph of $y=r(x)$.



[0, 60] by [−50, 200]

75. (a) Since $x=\tan\theta$, we have

$\frac{dx}{dt}=(\sec^2\theta)\frac{d\theta}{dt}=-0.6\sec^2\theta$. At point A, we have $\theta=0$ and $\frac{dx}{dt}=-0.6\sec^2 0=-0.6$ km/sec.

(b) $0.6\ \frac{\text{rad}}{\text{sec}}\cdot\frac{1\text{ revolution}}{2\pi\text{ rad}}\cdot\frac{60\text{sec}}{1\text{min}}=\frac{18}{\pi}$ revolutions per minute or approximately 5.73 revolutions per minute.

76. Let $f(x)=\sin(x-\sin x)$. Then

$f'(x)=\cos(x-\sin x)\frac{d}{dx}(x-\sin x)$

$=\cos(x-\sin x)(1-\cos x)$. This derivative is zero when $\cos(x-\sin x)=0$ (which we need not solve) or when $\cos x=1$, which occurs at $x=2k\pi$ for integers k. For each of these values, $f(x)=f(2k\pi)=\sin(2k\pi-\sin 2k\pi)=\sin(2k\pi-0)=0$. Thus, $f(x)=f'(x)=0$ for $x=2k\pi$, which means that the graph has a horizontal tangent at each of these values of x.

77. $y'(r)=\frac{d}{dr}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right)=\left(\frac{1}{2l}\sqrt{\frac{T}{\pi d}}\right)\frac{d}{dr}\left(\frac{1}{r}\right)=-\frac{1}{2r^2 l}\sqrt{\frac{T}{\pi d}}$

$y'(l)=\frac{d}{dr}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right)=\left(\frac{1}{2r}\sqrt{\frac{T}{\pi d}}\right)\frac{d}{dr}\left(\frac{1}{l}\right)=-\frac{1}{2r^2 l}\sqrt{\frac{T}{\pi d}}$

$y'(d)=\frac{d}{dd}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right)=\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi}}\right)\frac{d}{dd}(d^{-1/2})$

$=\frac{1}{2rl}\sqrt{\frac{T}{\pi}}\left(-\frac{1}{2}d^{-3/2}\right)=-\frac{1}{4rl}\sqrt{\frac{T}{\pi d^3}}$

$y'(T)=\frac{d}{dT}\left(\frac{1}{2rl}\sqrt{\frac{T}{\pi d}}\right)=\left(\frac{1}{2rl}\sqrt{\frac{1}{\pi d}}\right)\frac{d}{dt}(\sqrt{T})$

$=\frac{1}{2rl}\sqrt{\frac{1}{\pi d}}\left(\frac{1}{2\sqrt{T}}\right)=\frac{1}{4rk\sqrt{\pi dT}}$

Since $y'(r)<0, y'(l)<0$, and $y'(d)<0$, increasing r, l, or d would decrease the frequency. Since $y'(T)>0$, increasing T would increase the frequency.

78. **(a)** $P(0)=\frac{200}{1+e^5}\approx 1$ student

(b) $\lim_{h\to\infty} P(t)=\lim_{h\to\infty}\frac{200}{1+e^{5-t}}=\frac{200}{1}=200$ students

(c) $P'(t)=\frac{d}{dt}200(1+e^{5-t})^{-1}$

$=1200(1+e^{5-t})^{-2}(e^{5-t})(-1)$

$=\frac{200e^{5-t}}{(1+e^{5-t})^2}$

$P''(t)=\frac{(1+e^{5-t})^2(200e^{5-t})(-1)-(200e^{5-t})(2)(1+e^{5-t})(e^{5-t})(-1)}{(1+e^{5-t})^4}$

$=\frac{(1+e^{5-t})(-200e^{5-t})+400(e^{5-t})^2}{(1+e^{5-t})^3}$

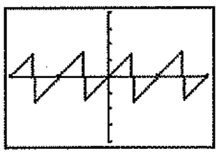
$=\frac{(200e^{5-t})(e^{5-t}-1)}{(1+e^{5-t})^3}$

Since $P''=0$ when $t=5$, the critical point of $y=P'(t)$ occurs at $t=5$. To confirm that this corresponds to the maximum value of $P'(t)$, note that $P''(t)>0$ for $t<5$ and $P''(t)<0$ for $t>5$. The maximum rate occurs at $t=5$, and this rate is

$P'(5)=\frac{200e^0}{(1+e^0)^2}+\frac{200}{2^2}=50$ students per day.

Note: This problem can also be solved graphically.

79.



$[-\pi, \pi]$ by $[-4, 4]$

(a) $x\neq k\frac{\pi}{4}$, where k is an odd integer

(b) $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$

(c) Where it's not defined, at $x=k\frac{\pi}{4}$, k an odd integer

(d) It has period $\frac{\pi}{2}$ and continues to repeat the pattern seen in this window.

80. Use implicit differentiation.

$x^2-y^2=1$

$\frac{d}{dx}(x^2)-\frac{d}{dx}(y^2)=\frac{d}{dx}(1)$

$2x-2yy'=0$

$y'=\frac{2x}{2y}=\frac{x}{y}$

$y''=\frac{d}{dx}\frac{x}{y}$

$=\frac{(y)(1)-(x)(y')}{y^2}$

$=\frac{y-x\left(\frac{x}{y}\right)}{y^2}$

$=\frac{y^2-x^2}{y^3}$

$=-\frac{1}{y^3}$

(since the given equation is $x^2-y^2=1$)

At $(2,\sqrt{3})$, $\frac{d^2y}{dx^2}=-\frac{1}{y^3}=-\frac{1}{(\sqrt{3})^3}=-\frac{1}{3\sqrt{3}}$.

81. **(a)** $v(t)=\frac{ds}{dt}=\frac{d}{dt}(t^3-2t+3)$

$v(t)=3t^2-2$

(b) $a(t)=\frac{dv}{dt}=\frac{d}{dt}=(3t^2-2)$

$a(t)=6t$

(c)

$v(t)=3t^2-2=0$

$t^2=\frac{2}{3}$

$t=\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{6}}{3}$

$t^2=\frac{2}{3}$

$t=\frac{\sqrt{2}}{\sqrt{3}}=\frac{\sqrt{6}}{3}$

(d) $v(t)=3t^2-2<0$

$3t^2<2$

$t<\frac{\sqrt{6}}{3}$, and $t>0$

$0<t<\frac{\sqrt{6}}{3}$

81. Continued

(e) $v(t) = 3t^2 - 2 > 0$

$3t^2 > 2$

$t > \dfrac{\sqrt{6}}{2}$

82. (a) $\dfrac{d}{dx}e^u = e^u \dfrac{du}{dx}$ where $u = x$

$\dfrac{d}{dx}\dfrac{e^x + e^{-x}}{2} = \dfrac{e^x - e^{-x}}{2}$

(b) $\dfrac{d}{dx}\dfrac{e^x - e^{-x}}{2} = \dfrac{e^x + e^{-x}}{2}$

(c) $y(1) = \dfrac{e^1 + e^{-1}}{2} = 1.543$

$y'(1) = \dfrac{e^1 - e^{-1}}{2} = 1.175$

$y = 1.175(x-1) + 1.543$

$y = 1.175x + 0.368$

(d) $m_2 = -\dfrac{1}{m_1} = -\dfrac{1}{1.175} = -0.851$

$y = -0.851(x-1) + 1.543$

$y = -0.851x + 2.394$

(e) $y' = 0 = \dfrac{e^x - e^{-x}}{2}$

$0 = e^x - e^{-x}$

$e^x = e^{-x}$

$x = -x$ or $x = 0$

83. (a) $1 - x^2 > 0$

$x^2 > 1, -1 < x < 1$

(b) $f'(x) = \dfrac{d}{dx}\ln(1 - x^2)$ $\qquad u = 1 - x^2$

$\dfrac{d}{dx}\ln(u) = \dfrac{1}{u}\dfrac{du}{dx}$ $\qquad \dfrac{du}{dx} = -2x$

$= \dfrac{-2x}{(1 - x^2)}$

(c) $1 - x^2 > 0, -1 < x < 1$

(d) $y'\left(\dfrac{1}{2}\right) = \dfrac{-2\left(\dfrac{1}{2}\right)}{1 - \left(\dfrac{1}{2}\right)^2} = -\dfrac{1}{3/4} = -4/3$

Chapter 4
Applications of Derivatives

Section 4.1 Extreme Values of Functions (pp. 187–195)

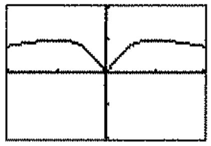
Exploration 1 Finding Extreme Values

1. From the graph we can see that there are three critical points: $x = -1, 0, 1$.

Critical point values: $f(-1) = 0.5$, $f(0) = 0$, $f(1) = 0.5$

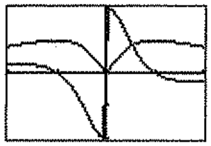
Endpoint values: $f(-2) = 0.4$, $f(2) = 0.4$

Thus f has absolute maximum value of 0.5 at $x = -1$ and $x = 1$, absolute minimum value of 0 at $x = 0$, and local minimum value of 0.4 at $x = -2$ and $x = 2$.



$[-2, 2]$ by $[-1, 1]$

2. The graph of f' has zeros at $x = -1$ and $x = 1$ where the graph of f has local extreme values. The graph of f' is not defined at $x = 0$, another extreme value of the graph of f.



$[-2, 2]$ by $[-1, 1]$

3. Using the chain rule and $\dfrac{d}{dx}(|x|) = \dfrac{|x|}{x}$, we find

$\dfrac{df}{dx} = \dfrac{|x|}{x} \cdot \dfrac{1 - x^2}{(x^2 + 1)^2}$.

Quick Review 4.1

1. $f'(x) = \dfrac{1}{2\sqrt{4-x}} \cdot \dfrac{d}{dx}(4 - x) = \dfrac{-1}{2\sqrt{4-x}}$

2. $f'(x) = \dfrac{d}{dx}2(9 - x^2)^{-1/2} = -(9 - x^2)^{-3/2} \cdot \dfrac{d}{dx}(9 - x^2)$

$= -(9 - x^2)^{-3/2}(-2x) = \dfrac{2x}{(9 - x^2)^{3/2}}$

3. $g'(x) = -\sin(\ln x) \cdot \dfrac{d}{dx}\ln x = -\dfrac{\sin(\ln x)}{x}$

4. $h'(x) = e^{2x} \cdot \dfrac{d}{dx}2x = 2e^{2x}$

5. Graph (c), since this is the only graph that has positive slope at c.

6. Graph (b), since this is the only graph that represents a differentiable function at a and b and has negative slope at c.

7. Graph (d), since this is the only graph representing a function that is differentiable at b but not at a.

8. Graph (a), since this is the only graph that represents a function that is not differentiable at a or b.

9. As $x \to 3^-$, $\sqrt{9 - x^2} \to 0^+$. Therefore, $\lim\limits_{x \to 3^-} f(x) = \infty$.

10. As $x \to 3^+$, $\sqrt{9 - x^2} \to 0^+$. Therefore, $\lim\limits_{x \to -3^+} f(x) = \infty$.

11. (a) $\frac{d}{dx}(x^3 - 2x) = 3x^2 - 2$

$f'(1) = 3(1)^2 - 2 = 1$

(b) $\frac{d}{dx}(x+2) = 1$

$f'(3) = 1$

(c) Left-hand derivative:

$$\lim_{h\to 0^-} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^-} \frac{[(2+h)^3 - 2(2+h)] - 4}{h}$$
$$= \lim_{h\to 0^-} \frac{h^3 + 6h^2 + 10h}{h}$$
$$= \lim_{h\to 0^-} (h^2 + 6h + 10)$$
$$= 10$$

Right-hand derivative:

$$\lim_{h\to 0^+} \frac{f(2+h)-f(2)}{h} = \lim_{h\to 0^+} \frac{[(2+h)+2]-4}{h}$$
$$= \lim_{h\to 0^+} \frac{h}{h}$$
$$= \lim_{h\to 0^+} 1$$
$$= 1$$

Since the left-and right-hand derivatives are not equal, $f'(2)$ is underfined.

12. (a) The domain is $x \neq 2$. (See the solution for 11.(c)).

(b) $f'(x) = \begin{cases} 3x^2 - 2, & x < 2 \\ 1, & x > 2 \end{cases}$

Section 4.1 Exercises

1. Minima at (–2, 0) and (2, 0), maximum at (0, 2)

2. Local minimum at (–1, 0), local maximum at (1, 0)

3. Maximum at (0, 5) Note that there is no minimum since the endpoint (2, 0) is excluded from the graph.

4. Local maximum at (–3, 0), local minimum at (2, 0), maximum at (1, 2), minimum at (0, –1)

5. Maximum at $x = b$, minimum at $x = c_2$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.

6. Maximum at $x = c$, minimum at $x = b$;
The Extreme Value Theorem applies because f is continuous on $[a, b]$, so both the maximum and minimum exist.

7. Maximum at $x = c$, no minimum;
The Extreme Value Theorem does not apply, because the function is not defined on a closed interval.

8. No maximum, no minimum;
The Extreme Value Theorem does not apply, because the function is not continuous or defined on a closed interval.

9. Maximum at $x = c$, minimum at $x = a$;
The Extreme Value Theorem does not apply, because the function is not continuous.

10. Maximum at $x = a$, minimum at $x = c$;
The Extreme Value Theorem does not apply since the function is not continuous.

11. The first derivative $f'(x) = -\frac{1}{x^2} + \frac{1}{x}$ has a zero at $x = 1$.

Critical point value: $f(1) = 1 + \ln 1 = 1$

Endpoint values: $f(0.5) = 2 + \ln 0.5 \approx 1.307$

$$f(4) = \frac{1}{4} + \ln 4 \approx 1.636$$

Maximum value is $\frac{1}{4} + \ln 4$ at $x = 4$;

minimum value is 1 at $x = 1$;

local maximum at $\left(\frac{1}{2}, 2 - \ln 2\right)$

12. The first derivative $g'(x) = -e^{-x}$ has no zeros, so we need only consider the endpoints.

$g(-1) = e^{-(-1)} = e \qquad g(1) = e^{-1} = \frac{1}{e}$

Maximum value is e at $x = -1$;

minimum value is $\frac{1}{e}$ at $x = 1$.

13. The first derivative $h'(x) = \frac{1}{x+1}$ has no zeros, so we need only consider the endpoints.

$h(0) = \ln 1 = 0 \qquad h(3) = \ln 4$

Maximum value is ln 4 at $x = 3$;

minimum value is 0 at $x = 0$.

14. The first derivative $k'(x) = -2xe^{-x^2}$ has a zero at $x = 0$. Since the domain has no endpoints, any extreme value must occur at $x = 0$. Since $k(0) = e^{-0^2} = 1$ and $\lim_{x\to\pm\infty} k(x) = 0$, the maximum value is 1 at $x = 0$.

15. The first derivative $f'(x) = \cos\left(x + \frac{\pi}{4}\right)$, has zeros at $x = \frac{\pi}{4}$, $x = \frac{5\pi}{4}$.

Critical point values: $x = \frac{\pi}{4} \quad f(x) = 1$

$x = \frac{5\pi}{4} \quad f(x) = -1$

Endpoint values: $x = 0 \quad f(x) = \frac{1}{\sqrt{2}}$

$x = \frac{7\pi}{4} \quad f(x) = 0$

Maximum value is 1 at $x = \frac{\pi}{4}$;

minimum value is –1 at $x = \frac{5\pi}{4}$;