67. Let C = circumference, r = radius, S = surface area, and V = volume.

(a) Since $C = 2\pi r$, we have $\frac{dC}{dr} = 2\pi$ and so $dC = 2\pi\ dr$.

Therefore, $\left|\frac{dC}{C}\right| = \left|\frac{2\pi dr}{2\pi r}\right| = \left|\frac{dr}{r}\right| < \frac{0.4\text{ cm}}{10\text{ cm}} = 0.04$ The calculated radius will be within approximately 4% of the correct radius.

(b) Since $S = 4\pi r^2$, we have $\frac{dS}{dr} = 8\pi r$ and so $dS = 8\pi r\ dr$. Therefore,

$\left|\frac{dS}{S}\right| = \left|\frac{8\pi r\ dr}{4\pi r^2}\right| = \left|\frac{2\ dr}{r}\right| \le 2(0.04) = 0.08$. The calculated surface area will be within approximately 8% of the correct surface area.

(c) Since $V = \frac{4}{3}\pi r^3$, we have $\frac{dV}{dr} = 4\pi r^2$ and so $dV = 4\pi r^2\ dr$. Therefore

$$\left|\frac{dV}{V}\right| = \left|\frac{4\pi r^2\ dr}{\frac{4}{3}\pi r^3}\right| = \left|\frac{3\ dr}{r}\right| \le 3(0.04) = 0.12.$$

The calculated volume will be within approximately 12% of the correct volume.

68. By similar triangles, we have $\frac{a}{6} = \frac{a+20}{h}$, which gives $ah = 6a + 120$, or $h = 6 + 120a^{-1}$ The height of the lamp post is approximately $6 + 120(15)^{-1} = 14$ ft. The estimated error in measuring a was $|da| \le 1$ in. $= \frac{1}{12}$ ft. Since $\frac{dh}{da} = -120a^{-2}$, we have

$|dh| = \left|-120a^{-2}\ da\right| \le 120(15)^{-2}\left(\frac{1}{12}\right) = \frac{2}{45}$ ft, so the estimated possible error is $\pm\frac{2}{45}$ ft or $\pm\frac{8}{15}$ in.

69. $\frac{dy}{dx} = 2\sin x\cos x - 3$. Since $\sin x$ and $\cos x$ are both between 1 and -1, the value of $2\sin x\cos x$ is never greater than 2. Therefore, $\frac{dy}{dx} \le 2 - 3 = -1$ for all values of x.

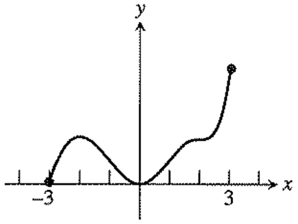
Since $\frac{dy}{dx}$ is always negative, the function decreases on every interval.

70. (a) f has a relative maximum at $x = -2$. This is where $f'(x) = 0$, causing f' to go from positive to negative.

(b) f has a relative minimum at $x = 0$. This is where $f'(x) = 0$, causing f' to go from negative to positive.

(c) The graph of f is concave up on $(-1, 1)$ and on $(2, 3)$. These are the intervals on which the derivatives of f are increasing.

(d)



71. (a) $A = \pi r^2$

$$\frac{dA}{dt} = 2\pi r dr$$

$$\frac{dA}{dt} = 2\pi(2)\left(\frac{1}{3}\right) = \frac{4}{3}\pi\frac{\text{in.}^2}{\text{sec}}$$

(b) $dA = dV$

$$\frac{4}{3}\pi = \frac{1}{3}\pi r^2 dh$$

$$\frac{4}{3}\pi = \frac{1}{3}\pi(2)^2 dh$$

$$\frac{dh}{dt} = 1\frac{\text{in.}}{\text{sec}}$$

(c) $\frac{dA}{dh} = \frac{\frac{4}{3}\pi}{1} = \frac{4}{3}\pi\frac{\text{in.}^2}{\text{in.}}$

72. (a)

$$2a + 4b = 60$$

$$b = 15 - 2a$$

$$V = \pi a^2 b = \pi a^2(15 - 2a)$$

$$\frac{dV}{da} = 30\pi a - \frac{3\pi a^2}{2}$$

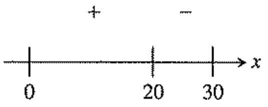
$$30\pi a = \frac{3\pi a^2}{2}$$

$$a = 20$$

$$2(20) + 4b = 60$$

$$b = 5$$

(b) The sign graph for the derivative $\frac{dV}{da} = \frac{3\pi a}{2}(20 - a)$ on the interval (0,30) is as follows:



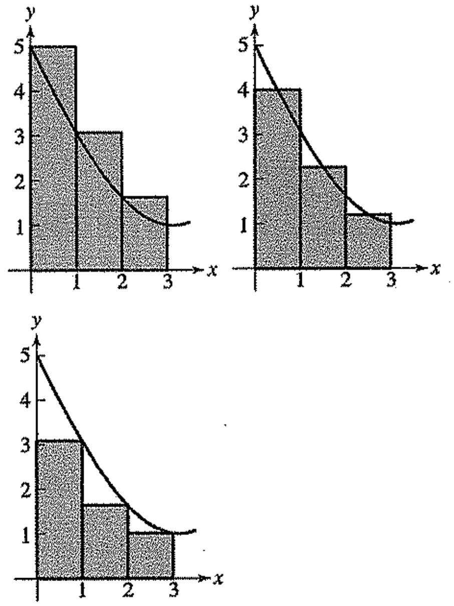
By the First Derivative Test, there is a maximum at $x = 20$.

Chapter 5
The Definite Integral

Section 5.1 Estimating with Finite Sums (pp. 263-273)

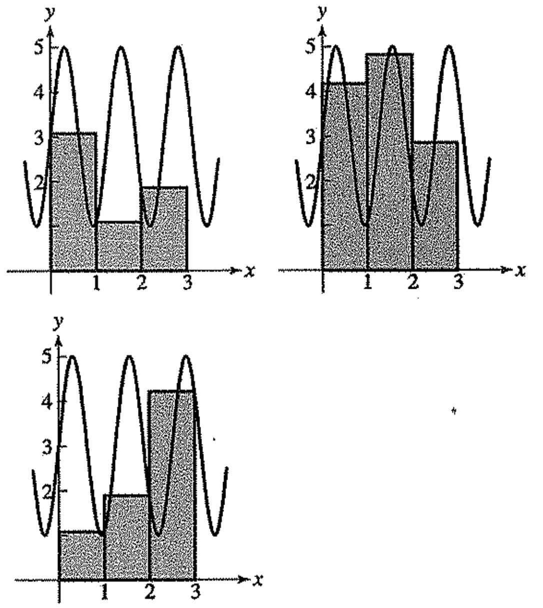
Exploration 1 Which RAM is the Biggest?

1.



LRAM > MRAM > RRAM

2.



MRAM > RRAM > LRAM

3. RRAM > MRAM > LRAM, because the heights of the rectangles increase as you move toward the right under an increasing function.

4. LRAM > MRAM > RRAM, because the heights of the rectangles decrease as you move toward the right under a decreasing function.

Quick Review 5.1

1. $80 \text{ mph} \cdot 5 \text{ hr} = 400 \text{ mi}$

2. $48 \text{ mph} \cdot 3 \text{ hr} = 144 \text{ mi}$

3. $10 \text{ ft/sec}^2 \cdot 10 \text{ sec} = 100 \text{ ft/sec}$

$$100 \text{ ft/sec} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \cdot \frac{3600 \text{ sec}}{1 \text{ h}} \approx 68.18 \text{ mph}$$

4. $300{,}000 \text{ km / sec} \cdot \frac{3600 \text{ sec}}{1 \text{ hr}} \cdot \frac{24 \text{ hr}}{1 \text{ day}} \cdot \frac{365 \text{ days}}{1 \text{ yr}} \cdot 1 \text{ yr}$

$\approx 9.46 \times 10^{12} \text{ km}$

5. $(6 \text{ mph})(3 \text{ h}) + (5 \text{ mph})(2 \text{ h}) = 18 \text{ mi} + 10 \text{ mi} = 28 \text{ mi}$

6. $20 \text{ gal/min} \cdot 1 \text{ h} \cdot \frac{60 \text{ min}}{1 \text{ h}} = 1200 \text{ gal}$

7. $(-1°\text{C/h})(12 \text{ h}) + (1.5°\text{C})(6 \text{ h}) = -3°\text{C}$

8. $300 \text{ ft}^3/\text{sec} \cdot \frac{3600 \text{ sec}}{1 \text{ h}} \cdot \frac{24 \text{ h}}{1 \text{ day}} \cdot 1 \text{ day} = 25{,}920{,}000 \text{ ft}^3$

9. $350 \text{ people/mi}^2 \cdot 50 \text{ mi}^2 = 17{,}500 \text{ people}$

10. $70 \text{ times/sec} \cdot \frac{3600 \text{ sec}}{1 \text{ h}} \cdot 1 \text{ h} \cdot 0.7 = 176{,}400 \text{ times}$

Section 5.1 Exercises

1. Since $v(t) = 5$ is a strait line, compute the area under the curve.
$x = (t)\ v(t) = (4)(5) = 20$

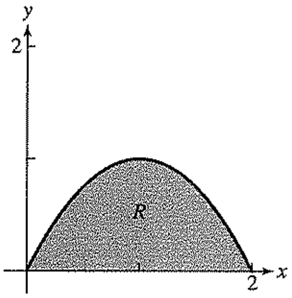
2. Since $v(t) = 2t + 1$ creates a trapezoid with the x-axis, compute the area of the curve under the trapezoid.

$$A = \frac{h}{2}(a + b)$$
$$a = t = 0 = v(0) = 2(0) + 1 = 1$$
$$b = t = 4 = v(4) = 2(4) + 1 = 9$$
$$h = 4$$
$$A = \frac{4}{2}(9 + 1) = 20$$

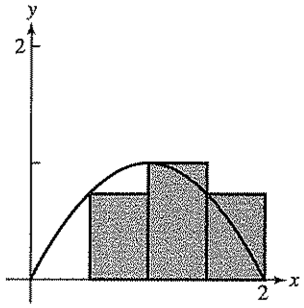
3. Each rectangle has base 1. The height of each rectangle is Found by using the points $t = (0.5, 1.5, 2.5, 3.5)$ in the equation $v(t)\ t^2 + 1$. The area under the curve is approximately $1\left(\frac{5}{4} + \frac{13}{4} + \frac{29}{4} + \frac{53}{4}\right) = 25$, so the particle is close to $x = 25$.

4. Each rectangle has base 1. The height of each rectangle is found by using the points $y = (0.5, 1.5, 2.5, 3.5, 4.5)$ in the equation $v(t) = t^2 + 1$. The area under the curve is approximately $1\left(\frac{5}{4} + \frac{13}{4} + \frac{29}{4} + \frac{53}{4} + \frac{85}{4}\right) = 46.25$, so the particle is close to $x = 46.25$.

5. (a)



(b)

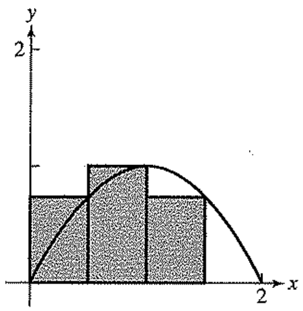


$\Delta x = \frac{1}{2}$

LRAM: $[2(0)-(0)^2]\left(\frac{1}{2}\right)+\left[2\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^2\right]\left(\frac{1}{2}\right)$

$+[2(1)-(1)^2]\left(\frac{1}{2}\right)+\left[2\left(\frac{3}{2}\right)-\left(\frac{3}{2}\right)^2\right]\left(\frac{1}{2}\right)=\frac{5}{4}=1.25$

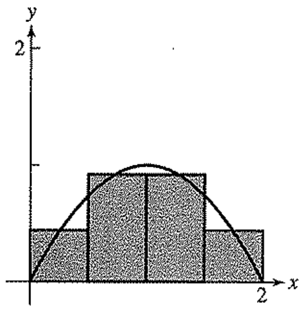
6. (a)



RRAM: $\left[2\left(\frac{1}{2}\right)-\left(\frac{1}{2}\right)^2\right]\left(\frac{1}{2}\right)+[2(1)-(1)^2]\left(\frac{1}{2}\right)$

$+\left[2\left(\frac{3}{2}\right)-\left(\frac{3}{2}\right)^2\right]\left(\frac{1}{2}\right)+[2(2)-(2)^2]\left(\frac{1}{2}\right)=\frac{5}{4}=1.25$

(b)



MRAM: $\left[2\left(\frac{1}{4}\right)-\left(\frac{1}{4}\right)^2\right]\left(\frac{1}{2}\right)+\left[2\left(\frac{3}{4}\right)-\left(\frac{3}{4}\right)^2\right]\left(\frac{1}{2}\right)$

$+\left[2\left(\frac{5}{4}\right)-\left(\frac{5}{4}\right)^2\right]\left(\frac{1}{2}\right)+\left[2\left(\frac{7}{4}\right)-\left(\frac{7}{4}\right)^2\right]\left(\frac{1}{2}\right)=\frac{11}{8}=1.375$

7.

n	LRAM$_n$	MRAM$_n$	RRAM$_n$
10	1.32	1.34	1.32
50	1.3328	1.3336	1.3328
100	1.3332	1.3334	1.3332
500	1.333328	1.333336	1.333328

8. The area is $1.\overline{333}=\frac{4}{3}$.

9.

n	LRAM$_n$	MRAM$_n$	RRAM$_n$
10	12.645	13.4775	14.445
50	13.3218	13.4991	13.6818
100	13.41045	13.499775	13.59045
500	13.482018	13.499991	13.518018

Estimate the area to be 13.5.

10.

n	LRAM$_n$	MRAM$_n$	RRAM$_n$
10	1.16823	1.09714	1.03490
50	1.11206	1.09855	1.08540
100	1.10531	1.09860	1.09198
500	1.09995	1.09861	1.09728
1000	1.09928	1.09861	1.09795

Estimate the area to be 1.0986.

11.

n	LRAM$_n$	MRAM$_n$	RRAM$_n$
10	0.98001	0.88220	0.78367
50	0.90171	0.88209	0.86244
100	0.89190	0.88208	0.87226
500	0.88404	0.88208	0.88012
1000	0.88306	0.88208	0.88110

Estimate the area to be 0.8821.

12.

n	LRAM$_n$	MRAM$_n$	RRAM$_n$
10	1.98352	2.00825	1.98352
50	1.99934	2.00033	1.99934
100	1.99984	2.00008	1.99984
500	1.99999	2.00000	1.99999

Estimate the area to be 2.

13. Use $f(x)=\sqrt{25-x^2}$ and approximate the volume using $\pi r^2 h=\pi(\sqrt{25-n_i^2})^2\Delta x$, so for the MRAM program, use $\pi(25-x^2)$ on the interval $[-5,5]$.

n	MRAM
10	526.21677
20	524.25327
40	523.76240
80	523.63968
160	523.60900

14. $V=\frac{4}{3}\pi(5)^3=\frac{500\pi}{3}\approx 523.59878$

n	error	% error
10	2.61799	0.5
20	0.65450	0.125
40	0.16362	0.0312
80	0.04091	0.0078
160	0.01023	0.0020

15. LRAM:

Area

$\approx f(2)\bullet 2+f(4)\bullet 2+f(6)\bullet 2+\cdots+f(22)\bullet 2$

$=2\bullet(0+0.6+1.4+\cdots+0.5)$

$=44.8$ (mg/L) • sec

RRAM:

Area

$\approx f(4)\bullet 2+f(6)\bullet 2+f(8)\bullet 2+\cdots+f(24)\bullet 2$

$=2(0.6+1.4+2.7+\cdots+0)$

$=44.8$ (mg/L) • sec

Patient's cardiac output:

$$\frac{5\text{ mg}}{44.8\text{ (mg/L)}\bullet\text{sec}}\bullet\frac{60\text{ sec}}{1\text{ min}}\approx 6.7\text{ L/min}$$

Note that estimates for the area may vary.

16. (a) LRAM: $1\bullet(0+12+22+10+5+13+11+6+2+6)=87$ in. $=7.25$ ft

(b) RRAM: $1\bullet(12+22+10+5+13+11+6+2+6+0)=87$ in. $=7.25$ ft

17. 5 min = 300 sec

(a) LRAM: $300\bullet(1+1.2+1.7+\cdots+1.2)=5220$ m

(b) RRAM: $300\bullet(1.2+1.7+2.0+\cdots+0)=4920$ m

18. LRAM: $10\bullet(0+44+15+\cdots+30)=3490$ ft

RRAM: $10\bullet(44+15+35+\cdots+35)=3840$ ft

$$\text{Average}=\frac{3490\text{ ft}+3840\text{ ft}}{2}=3665\text{ ft}$$

19. (a) LRAM: $0.001(0+40+62+\cdots+137)=0.898$ mi

RRAM: $0.001(40+62+82+\cdots+142)=1.04$ mi

Average = 0.969 mi

(b) The halfway point is 0.4845 mi. The average of LRAM and RRAM is 0.4460 at 0.006 h and 0.5665 at 0.007 h. Estimate that it took 0.006 h = 21.6 sec. The car was going 116 mph.

20. (a) Use LRAM with $\pi(16-x^2)$.

$S_8\approx 146.08406$

S_8 is an overestimate because each rectangle is below the curve.

(b) $\frac{|V-S_8|}{V}\approx 0.09=9\%$

21. (a) Use RRAM with $\pi(16-x^2)$.

$S_8\approx 120.95132$

S_8 is an underestimate because each rectangle is below the curve.

(b) $\frac{|V-S_8|}{V}\approx 0.10=10\%$

22. (a) Use LRAM with $\pi(64-x^2)$ on the interval [4, 8], $n=8$.

$S\approx 372.27873\text{ m}^3$

(b) $\frac{|V-S_8|}{V}\approx 0.11=11\%$

23. **(a)** $(5)(6.0+8.2+9.1+\cdots+12.7)(30)\approx 15{,}465\text{ ft}^3$

(b) $(5)(8.2+9.1+9.9+\cdots+13.0)(30)\approx 16{,}515\text{ ft}^3$

24. Use LRAM with πx on the interval $[0,5]$, $n=5$.
$1(0+\pi+2\pi+3\pi+4\pi)=10\pi\approx 31.41593$

25. Use MRAM with πx on the interval $[0, 5]$, $n = 5$.

$$1\left(\frac{1}{2}\pi+\frac{3}{2}\pi+\frac{5}{2}\pi+\frac{7}{2}\pi+\frac{9}{2}\pi\right)=\frac{25}{2}\pi\approx 39.26991$$

26. **(a)** LRAM_5 :
$32.00+19.41+11.77+7.14+4.33=74.65\text{ ft/sec}$

(b) RRAM_5 :
$19.41+11.77+7.14+4.33+2.63=45.28\text{ ft/sec}$

(c) The upper estimates for speed are 32.00 ft/sec for the first sec, 32.00 + 19.41 = 51.41 ft/sec for the second sec, and 32.00 + 19.41 + 11.77 = 63.18 ft/sec for the third sec. Therefore, an upper estimate for the distance fallen is 32.00 + 51.41 + 63.18 = 146.59 ft.

27. **(a)** $400\text{ ft/sec}-(5\text{ sec})(32\text{ ft/sec}^2)=240\text{ ft/sec}$

(b) Use RRAM with $400-32x$ on $[0, 5]$, $n = 5$.
$368 + 336 + 304 + 272 + 240 = 1520\text{ ft}$

28. **(a)** Upper = 70 + 97 + 136 + 190 + 265 = 758 gal
Lower = 50 + 70 + 97 + 136 + 190 = 543 gal

(b) Upper = 70 + 97 + 136 + 190 + 265 + 369 + 516 + 720
= 2363 gal
Lower = 50 + 70 + 97 + 136 + 190 + 265 + 369 + 516
= 1693 gal

(c) 25,000 − 2363 = 22,637 gal

$$\frac{22{,}637}{720}\approx 31.44\text{ h (worst case)}$$

$$25{,}000-1693=23{,}307\text{ gal}$$

$$\frac{23{,}307}{720}\approx 32.37\text{ h (best case)}$$

29. **(a)** Since the release rate of pollutants is increasing, an upper estimate is given by using the data for the end of each month (right rectangles), assuming that new scrubbers were installed before the beginning of January. Upper estimate:
$30(0.20+0.25+0.27+0.34+0.45+0.52)$
≈ 60.9 tons of pollutants

A lower estimate is given by using the data for the end of the previous month (left rectangles). We have no data for the beginning of January, but we know that pollutants were released at the new-scrubber rate of 0.05 ton/day, so we may use this value.
Lower Estimate:
$30(0.05+0.20+0.25+0.27+0.34+0.45)$
≈ 46.8 tons of pollutants

(b) Using left rectangles, the amount of pollutants released by the end of October is
$30(0.05+0.20+0.25+0.27+0.34+0.45$
$+0.52+0.63+0.70+0.81)\approx 126.6$ tons

Therefore, a total of 125 tons will have been released into the atmosphere by the end of October.

30. The area of the region is the total number of units sold, in millions, over the 10-year period. The area units are (millions of units per year)(years) = (millions of units).

31. True. Because the graph rises from left to right, the left-hand rectangles will all lie under the curve.

32. False. For example, all three approximations are the same if the function is constant.

33. E. $y=4x-x^2=0$
$4x=x^2$
$x=0,4$
Use MRAM on the interval $[0, 4]$, $n = 4$.
$1(1.75 + 3.75 + 3.75 + 1.75) = 11$

34. D.

35. C.

$$\frac{\pi}{4}\left(\sin(0)+\sin\left(\frac{\pi}{4}\right)+\sin\left(\frac{\pi}{2}\right)+\sin\left(\frac{3\pi}{4}\right)\right)$$

$$\frac{\pi}{4}\left(0+\frac{\sqrt{2}}{2}+1+\frac{\sqrt{2}}{2}\right)$$

36. D.

37. **(a)** The diagonal of the square has length 2, so the side length is $\sqrt{2}$. Area $=(\sqrt{2})^2=2$

(b) Think of the octagon as a collection of 16 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{16}=\frac{\pi}{8}$.

$$\text{Area}=16\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{8}\right)\left(\cos\frac{\pi}{8}\right)$$

$$=4\sin\frac{\pi}{4}$$

$$=2\sqrt{2}\approx 2.828$$

(c) Think of the 16-gon as a collection of 32 right triangles with a hypotenuse of length 1 and an acute angle measuring $\frac{2\pi}{32}=\frac{\pi}{16}$.

$$\text{Area}=32\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{16}\right)\left(\cos\frac{\pi}{16}\right)$$

$$=8\sin\frac{\pi}{8}\approx 3.061$$

(d) Each area is less than the area of the circle, π. As n increases, the area approaches π.

38. The statement is false. We disprove it by presenting a counterexample, the function $f(x)=x^2$ over the interval $0\le x\le 1$, with $n=1$. $\text{MRAM}_1=1f(0.5)=0.25$

$$\frac{\text{LRAM}_1+\text{RRAM}_1}{2}=\frac{1f(0)+1f(1)}{2}$$

$$=\frac{0+1}{2}=0.5\ne\text{MRAM}_1$$

39. $\text{RRAM}_n f = (\Delta x)[f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + f(x_n)]$

$$= (\Delta x)[f(x_0) + f(x_1) + f(x_2) + \cdots + f(x_{n-1})] + (\Delta x)[f(x_n) - f(x_0)]$$

$$= \text{LRAM}_n f + (\Delta x)[f(x_n) - f(x_0)]$$

But $f(a) = f(b)$ by symmetry, so $f(x_n) - f(x_0) = 0$.
Therefore, $\text{RRAM}_n f = \text{LRAM}_n f$.

40. (a) Each of the isosceles triangles is made up of two right triangles having hypotenuse 1 and an acute angle measuring $\frac{2\pi}{2n} = \frac{\pi}{n}$. The area of each isosceles triangle is $A_T = 2\left(\frac{1}{2}\right)\left(\sin\frac{\pi}{n}\right)\left(\cos\frac{\pi}{n}\right) = \frac{1}{2}\sin\frac{2\pi}{n}$.

(b) The area of the polygon is

$A_P = nA_T = \frac{n}{2}\sin\frac{2\pi}{n}$, so

$\lim_{n\to\infty} A_P = \lim_{n\to\infty} \frac{n}{2}\sin\frac{2\pi}{n} = \pi$

(c) Multiply each area by r^2:

$A_T = \frac{1}{2}r^2\sin\frac{2\pi}{n}$

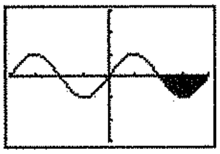
$A_P = \frac{n}{2}r^2\sin\frac{2\pi}{n}$

$\lim_{n\to\infty} A_P = \pi r^2$

Section 5.2 Definite Integrals (274–284)

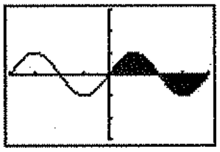
Exploration 1 Finding Integrals by Signed Areas

1. –2. (This is the same area as $\int_0^\pi \sin x\, dx$, but below the x-axis.)



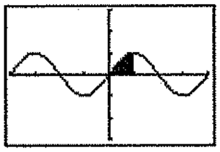
$[-2\pi, 2\pi]$ by $[-3, 3]$

2. 0. (The equal areas above and below the x- axis sum to zero.)



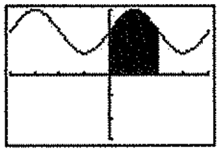
$[-2\pi, 2\pi]$ by $[-3, 3]$

3. 1. (This is half the area of $\int_0^\pi \sin x\, dx$.)



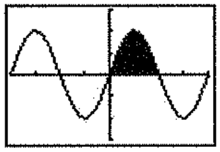
$[-2\pi, 2\pi]$ by $[-3, 3]$

4. $2\pi + 2$. The same area as $\int_0^\pi \sin x\, dx$ sits above a rectangle of area $\pi \times 2$.)



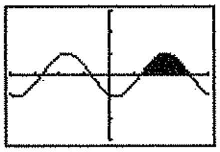
$[-2\pi, 2\pi]$ by $[-3, 3]$

5. 4. (Each rectangle in a typical Riemann sum is twice as tall as in $\int_0^\pi \sin x\, dx$.)



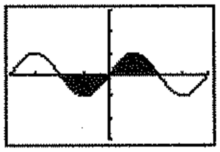
$[-2\pi, 2\pi]$ by $[-3, 3]$

6. 2. (This is the same region as in $\int_0^\pi \sin x\, dx$, translated 2 units to the right.)



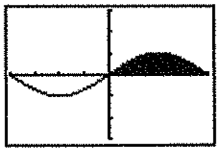
$[-2\pi, 2\pi]$ by $[-3, 3]$

7. 0. (The equal areas above and below the x-axis sum to zero.)



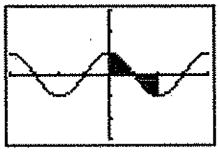
$[-2\pi, 2\pi]$ by $[-3, 3]$

8. 4. (Each rectangle in a typical Riemann sum is twice as wide as in $\int_0^\pi \sin x\, dx$.)



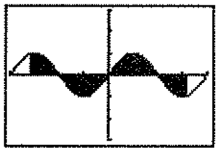
$[-2\pi, 2\pi]$ by $[-3, 3]$

9. 0. (The equal areas above and below the x-axis sum to zero.)



$[-2\pi, 2\pi]$ by $[-3, 3]$

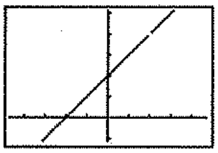
10. 0. (The equal areas above and below the x-axis sum to zero, since $\sin x$ is an odd function.)



$[-2\pi, 2\pi]$ by $[-3, 3]$

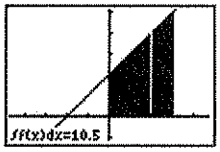
Exploration 2 More Discontinuous Integrands

1. The function has a removable discontinuity at $x = 2$.



$[-4.7, 4.7]$ by $[-1.1, 5.1]$

2. The thin strip above $x = 2$ has zero area, so the area under the curve is the same as $\int_0^3 (x+2)$, which is 10.5.



$[-4.7, 4.7]$ by $[-1.1, 5.1]$

3. The graph has jump discontinuities at all integer values, but the Riemann sums tend to the area of the shaded region shown. The area is the sum of the areas of 5 rectangles (one of them with height 0):

$\int_0^5 \text{int}(x)\,dx = 0+1+2+3+4 = 10.$

$[-2.7, 6.7]$ by $[-1.1, 5.1]$

Quick Review 5.2

1. $\sum_{n=1}^{5} n^2 = (1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2 = 55$

2. $\sum_{k=0}^{4} (3k-2) = [3(0)-2]+[3(1)-2]+[3(2)-2]$
$+[3(3)-2]+[3(4)-2] = 20$

3. $\sum_{j=0}^{4} 100(j+1)^2 = 100[(1)^2 + (2)^2 + (3)^2 + (4)^2 + (5)^2]$
$= 5500$

4. $\sum_{k=1}^{99} k$

5. $\sum_{k=0}^{25} 2k$

6. $\sum_{k=1}^{500} 3k^2$

7. $2\sum_{x=1}^{50} x^2 + 3\sum_{x=1}^{50} x = \sum_{x=1}^{50} (2x^2+3x)$

8. $\sum_{k=0}^{8} x^K + \sum_{k=9}^{20} x^k = \sum_{k=0}^{20} x^k$

9. $\sum_{k=0}^{n} (-1)^k = 0$ if n is odd.

10. $\sum_{k=0}^{n} (-1)^k = 1$ if n is even.

Section 5.2 Exercises

1. $\lim_{n\to\infty} \sum_{k=1}^{n} (c_k{}^2 \Delta x_k = \int_0^2 x^2 dx$ where n is any partition of $[0, 2]$.

2. $\lim_{n\to\infty} \sum_{k=1}^{n} (c_k{}^2 - 3c_k)\Delta x_k = \int_{-7}^{5} (x^2-3x)\,dx$ where n is any partition of $[-7,5]$.

3. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{c_k}\Delta x_k = \int_1^4 \frac{1}{x}\,dx$ where n is any partition of $[1, 4]$.

4. $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{1-c_k}\Delta x_k = \int_2^3 \frac{1}{1-x}\,dx$ where n is any partition of $[2, 3]$.

5. $\lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{4-c_k{}^2}\,\Delta x_k = \int_0^1 \sqrt{4-x^2}\,dx$ where n is any partition of $[0, 1]$.

6. $\lim_{n\to\infty} \sum_{k=1}^{n} (\sin^3 c_k)\Delta x_k = \int_{-\pi}^{\pi} \sin^3 x\,dx$ where n is any partition of $[-\pi,\pi]$.

7. $\int_{-2}^{1} 5\,dx = 5[1-(-2)] = 15$

8. $\int_3^7 (-20)\,dx = (-20)(7-3) = -80$

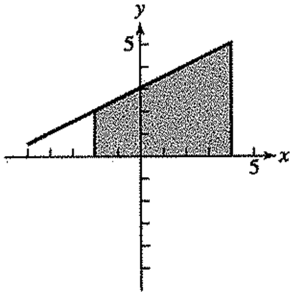
9. $\int_0^3 (-160)\,dt = (-160)(3-0) = -480$

10. $\int_{-4}^{-1} \frac{\pi}{2}\,d\theta = \frac{\pi}{2}[-1-(-4)] = \frac{3\pi}{2}$

11. $\int_{-2.1}^{3.4} 0.5\,ds = 0.5[3.4-(-2.1)] = 2.75$

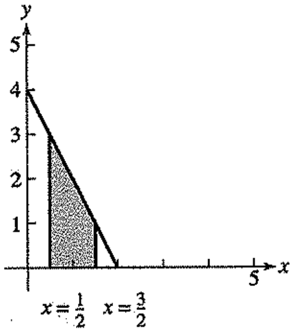
12. $\int_{\sqrt{2}}^{\sqrt{18}} \sqrt{2}\,dr = \sqrt{2}(\sqrt{18}-\sqrt{2}) = 4$

13. Graph the region under $y=\frac{x}{2}+3$ for $-2\le x\le 4$.



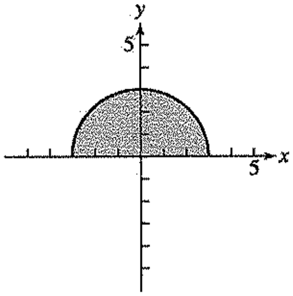
$$\int_{-2}^{4}\left(\frac{x}{2}+3\right)dx=\frac{1}{2}(6)(2+5)=21$$

14. Graph the region under $y=-2x+4$ for $\frac{1}{2}\le x\le\frac{3}{2}$.



$$\int_{1/2}^{3/2}(-2x+4)\,dx=\frac{1}{2}(1)(3+1)=2$$

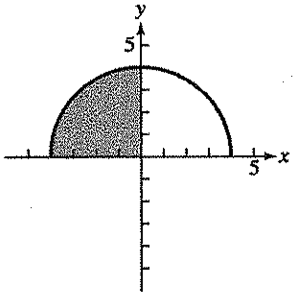
15. Graph the region under $y=\sqrt{9-x^2}$ for $-3\le x\le 3$.



This region is half of a circle radius 3.

$$\int_{-3}^{3}\sqrt{9-x^2}\,dx=\frac{1}{2}\pi(3)^2=\frac{9\pi}{2}$$

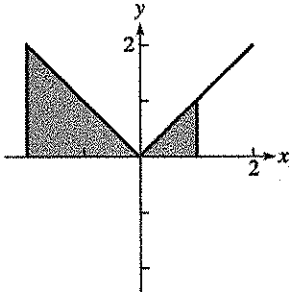
16. Graph the region under $y=\sqrt{16-x^2}$ for $-4\le x\le 0$.



The region is one quarter of a circle of radius 4.

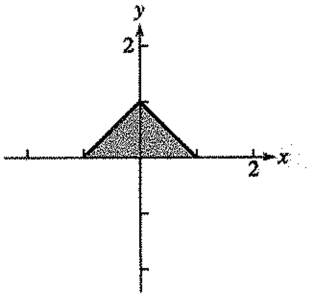
$$\int_{-4}^{0}\sqrt{16-x^2}\,dx=\frac{1}{4}\pi(4)^2=4\pi$$

17. Graph the region under $y=|x|$ for $-2\le x\le 1$.



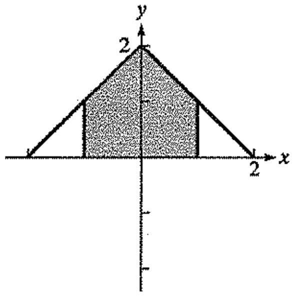
$$\int_{-2}^{1}|x|\,dx=\frac{1}{2}(2)(2)+\frac{1}{2}(1)(1)=\frac{5}{2}$$

18. Graph the region under $y=1-|x|$ for $-1\le x\le 1$.



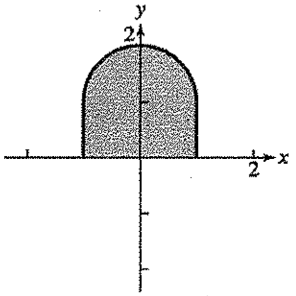
$$\int_{-1}^{1}(1-|x|)\,dx=\frac{1}{2}(2)(1)=1$$

19. Graph the region under $y = 2 - |x|$ for $-1 \le x \le 1$.



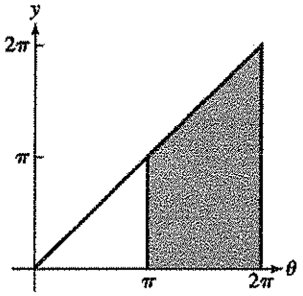
$$\int_{-1}^{1} (2 - |x|)\,dx = \frac{1}{2}(1)(1+2) + \frac{1}{2}(1)(1+2) = 3$$

20. Graph the region under $y = 1 + \sqrt{1 - x^2}$ for $-1 \le x \le 1$.



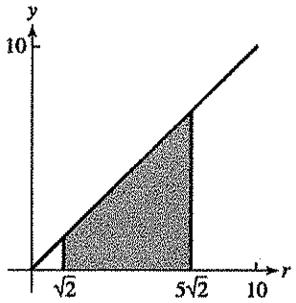
$$\int_{-1}^{1} (1 + \sqrt{1 - x^2})\,dx = (2)(1) + \frac{1}{2}\pi(1)^2 = 2 + \frac{\pi}{2}$$

21. Graph the region under $y = \theta$ for $\pi \le \theta \le 2\pi$



$$\int_{\pi}^{2\pi} \theta\,d\theta = \frac{1}{2}(2\pi - \pi)(2\pi + \pi) = \frac{3\pi^2}{2}$$

22. Graph the region under $y = r$ for $\sqrt{2} \le r \le 5\sqrt{2}$.



$$\int_{\sqrt{2}}^{5\sqrt{2}} r\,dr = \frac{1}{2}(5\sqrt{2} - \sqrt{2})(\sqrt{2} + 5\sqrt{2}) = 24$$

23. $\int_0^b x\,dx = \frac{1}{2}(b)(b) = \frac{1}{2}b^2$

24. $\int_0^b 4x\,dx = \frac{1}{2}(b)(4b) = 2b^2$

25. $\int_a^b 2s\,ds = \frac{1}{2}(b - a)(2b + 2a) = b^2 - a^2$

26. $\int_a^b 3t\,dt = \frac{1}{2}(b - a)(3b + 3a) = \frac{3}{2}(b^2 - a^2)$

27. $\int_a^{2a} x\,dx = \frac{1}{2}(2a - a)(2a + a) = \frac{3a^2}{2}$

28. $\int_a^{\sqrt{3a}} x\,dx = \frac{1}{2}(\sqrt{3a} - a)(\sqrt{3a} + a) = \frac{1}{2}(3a^2 - a^2) = a^2$

29. $\int_8^{11} 87\,dt = 87t\Big|_8^{11}$

$87(11) - 87(8) = 261$ miles

30. $\int_0^{60} 25\,dt = 25t\Big|_0^{60}$

$25(60) - 25(0) = 1500$ gallons

31. $\int_6^{7.5} 300\,dt = 300t\Big|_6^{7.5}$ calories

$300(7.5) - 300(6) = 450$

32. $\int_{8.5}^{11} 0.4\,dt = 0.4t\Big|_{8.5}^{11}$

$0.4(11) - 0.4(8.5) = 1$ liter

33. $\text{NINT}\left(\frac{x}{x^2 + 4}, x, 0, 5\right) \approx 0.9905$

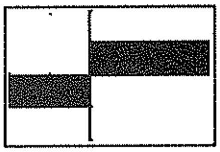
34. $3 + 2 \cdot \text{NINT}(\tan x, x, 0, \frac{}{3}) \approx 4.3863$

35. $\text{NINT}(4 - x^2, x, -2.2) \approx 10.6667$

36. $\text{NINT}(x^2 e^{-x},\ x, -1, 3) \approx 1.8719$

37. **(a)** The function has a discontinuity at $x = 0$.

(b)

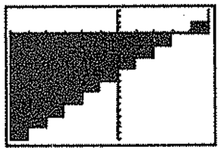


[−2, 3] by [−2, 2]

$$\int_{-2}^{3} \frac{x}{|x|}\,dx = -2 + 3 = 1$$

38. **(a)** The function has discontinuities at $x = -5, -4, -3, -2, -1, 0, 1, 2, 3, 4, 5$.

(b)

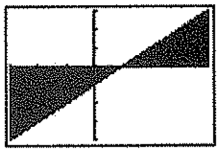


[−6, 5] by [−18, 4]

$$\int_{-6}^{5} 2\,\text{int}(x-3)\,dx = (-18) + (-16) + (-14) + (-12) + (-10) + (-8) + (-6) + (-4) + (-2) + 0 + 2 = -88$$

39. **(a)** The function has a discontinuity at $x = -1$.

(b)

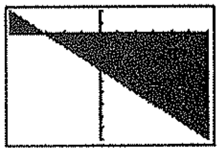


[−3, 4] by [−4, 3]

$$\int_{-3}^{4} \frac{x^2-1}{x+1}\,dx = -\frac{1}{2}(4)(4) + \frac{1}{2}(3)(3) = -\frac{7}{2}$$

40. **(a)** The function has a discontinuity at $x = 3$.

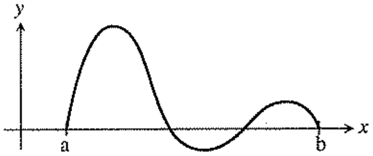
(b)



[−5, 6] by [−9, 2]

$$\int_{-5}^{6} \frac{9-x^2}{x-3}\,dx = \frac{1}{2}(2)(2) - \frac{1}{2}(9)(9) = -\frac{77}{2}$$

41. False. Consider the function in the graph below.



42. True. All the products in the Riemann sums are positive.

43. E. $\int_2^5 (f(x)+4)\,dx$

$$= \int_2^5 f(x)\,dx + \int_2^5 4\,dx$$
$$= 18 + 4x\Big|_2^5 = 30$$

44. D. $\int_{-4}^{4} (4 - |x|)\,dx$

$$= \int_{-4}^{4} 4\,dx + \int_0^4 x\,dx + \int_{-4}^{0} -x\,dx$$
$$= 4x\Big|_{-4}^{4} + \frac{x^2}{2}\Big|_0^4 - \frac{x^2}{2}\Big|_{-4}^{0} = 16$$

45. C.

46. A.

47. Observe that the graph of $f(x) = x^3$ is symmetric with respect to the origin. Hence the area above and below the x-axis is equal for $-1 \le x \le 1$.

$$\int_{-1}^{1} x^3\,dx = -(\text{area below } x\text{-axis}) + (\text{area above } x\text{-axis}) = 0$$

48. The graph of $f(x) = x^3 + 3$ is three units higher than the graph of $g(x) = x^3$. The extra area is $(3)(1) = 3$.

$$\int_0^1 (x^3 + 3)\,dx = \frac{1}{4} + 3 = \frac{13}{4}$$

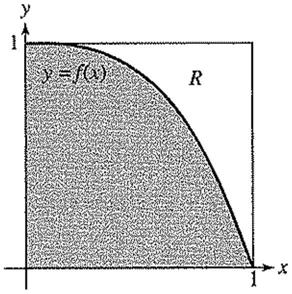
49. Observe that the region under the graph of $f(x) = (x-2)^3$ for $2 \le x \le 3$ is just the region under the graph of $g(x) = x^3$ for $0 \le x \le 1$ translated two units to the right.

$$\int_2^3 (x-2)^3\,dx = \int_0^1 x^3\,dx = \frac{1}{4}$$

50. Observe that the graph of $f(x) = |x|^3$ is symmetric with respect to the y-axis and the right half is the graph of $g(x) = x^3$.

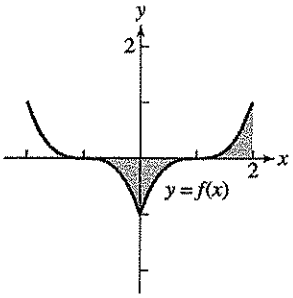
$$\int_{-1}^{1} |x|^3\,dx = 2\int_0^1 x^3\,dx = \frac{1}{2}$$

51. Observe from the graph below that the region under the graph $f(x) = 1 - x^3$ for $0 \le x \le 1$ cuts out a region R from the square identical to the region under the graph of $g(x) = x^3$ for $0 \le x \le 1$.



$$\int_0^1 (1 - x^3)\,dx = 1 - \int_0^1 x^3\,dx = 1 - \frac{1}{4} = \frac{3}{4}$$

52. Observe from the graph of $f(x)=(|x|-1)^3$ for $-1\le x\le 2$ that there are two regions below the x-axis and one region above the axis, each of whose area is equal to the area of the region under the graph of $g(x)=x^3$ for $0\le x\le 1$.



$$\int_{-1}^{2}(|x|-1)^3\,dx=2\left(-\frac{1}{4}\right)+\left(\frac{1}{4}\right)=-\frac{1}{4}$$

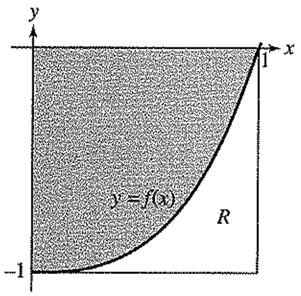
53. Observe that the graph of $f(x)=\left(\frac{x}{2}\right)^3$ for $0\le x\le 2$ is just a horizontal stretch of the graph of $g(x)=x^3$ for $0\le x\le 1$ by a factor of 2. Thus the area under $f(x)=\left(\frac{x}{2}\right)^3$ for $0\le x\le 2$ is twice the area under the graph of $g(x)=x^3$ for $0\le x\le 1$.

$$\int_0^2\left(\frac{x}{2}\right)^3 dx=2\int_0^1 x^3\,dx=\frac{1}{2}$$

54. Observe that the graph of $f(x)=x^3$ is symmetric with respect to the orgin. Hence the area above and below the x-axis is equal for $-8\le x\le 8$.

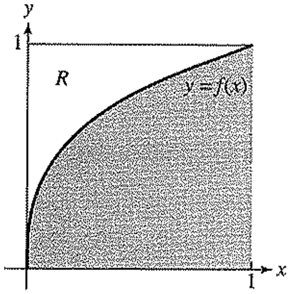
$$\int_{-8}^{8}x^3\,dx=-(\text{area below } x\text{-axis})+(\text{area above } x\text{-axis})=0$$

55. Observe from the graph below that the region between the graph of $f(x)=x^3-1$ and the x-axis for $0\le x\le 1$ cuts out a region R from the square identical to the region under the graph of $g(x)=x^3$ for $0\le x\le 1$.



$$\int_0^1(x^3-1)\,dx=-1+\frac{1}{4}=-\frac{3}{4}$$

56. Observe from the graph below that the region between the graph of $f(x)=\sqrt[3]{x}$ and the x-axis for $0\le x\le 1$ cuts out a region R from the square identical to the region under the graph of $g(x)=x^3$ for $0\le x\le 1$.



$$\int_0^1\sqrt[3]{x}\,dx=1-\frac{1}{4}=\frac{3}{4}$$

57. (a) As x approaches 0 from the right, $f(x)$ goes to ∞.

(b) Using right endpoints we have

$$\int_0^1\frac{1}{x^2}dx=\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{\left(\frac{k}{n}\right)^2}\right)\left(\frac{1}{n}\right)$$
$$=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{n}{k^2}$$
$$=\lim_{n\to\infty}n\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right).$$

Note that $n\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)>n$ and $n\to\infty$, so $n\left(1+\frac{1}{2^2}+\cdots+\frac{1}{n^2}\right)\to\infty$.

58. (a) $\Delta x=\frac{1}{n},\ x_k=\frac{k}{n}$

$$\text{RRAM}=\left(\frac{1}{n}\right)^2\bullet\frac{1}{n}+\left(\frac{2}{n}\right)^2\bullet\frac{1}{n}+\cdots+\left(\frac{n}{n}\right)^2\bullet\frac{1}{n}$$
$$=\sum_{k=1}^{n}\left(\left(\frac{k}{n}\right)^2\bullet\frac{1}{n}\right)$$

(b) $\sum_{k=1}^{n}\left(\left(\frac{k}{n}\right)^2\bullet\frac{1}{n}\right)=\sum_{k=1}^{n}\left(\frac{k^2}{n^3}\right)=\frac{1}{n^3}\sum_{k=1}^{n}k^2$

(c) $\frac{1}{n^3}\sum_{k=1}^{n}k^2=\frac{1}{n^3}\bullet\frac{n(n+1)(2n+1)}{6}=\frac{n(n+1)(2n+1)}{6n^3}$

(d) $\lim_{n\to\infty}\sum_{k=1}^{\infty}\left(\left(\frac{k}{n}\right)^2\bullet\frac{1}{n}\right)=\lim_{n\to\infty}\frac{n(n+1)(2n+1)}{6n^3}$

$$=\lim_{n\to\infty}\frac{2n^3+3n^2+n}{6n^3}$$
$$=\frac{2}{6}=\frac{1}{3}$$

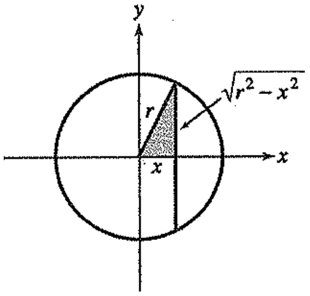
58. Continued

(e) Since $\int_0^1 x^2\,dx$ equals the limit of any Riemann sum over the interval [0, 1]as n approaches ∞, part (d) proves that $\int_0^1 x^2\,dx=\frac{1}{3}$.

Section 5.3 Definite Integrals and Antiderivatives (pp. 285–293)

Exploration 1 How Long is the Average Chord of a Circle?

1. The chord is twice as long as the leg of the right triangle in the first quadrant, which has length $\sqrt{r^2-x^2}$ by the Pythagorean Theorem.



2. Average value $=\dfrac{1}{r-(-r)}\displaystyle\int_{-r}^{r} 2\sqrt{r^2-x^2}\,dx.$

3. Average value $=\dfrac{2}{2r}\displaystyle\int_{-r}^{r}\sqrt{r^2-x^2}\,dx$

$$=\frac{1}{r}\cdot(\text{area of semicircle of radius } r)$$

$$=\frac{1}{r}\cdot\frac{\pi r^2}{2}$$

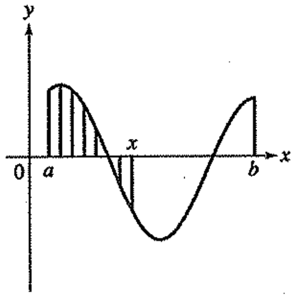
$$=\frac{\pi r}{2}$$

4. Although we only computed the average length of chords perpendicular to a particular diameter, the same computation applies to any diameter. The average length of a chord of a circle of radius r is $\frac{\pi r}{2}$.

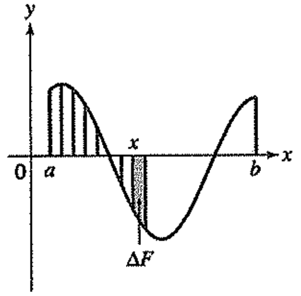
5. The function $y=2\sqrt{r^2-x^2}$ is continuous on $[-r, r]$, so the Mean Value Theorem applies and there is a c in $[a, b]$ so that $y(c)$ is the average value $\frac{\pi r}{2}$.

Exploration 2 Finding the Derivative of an Integral

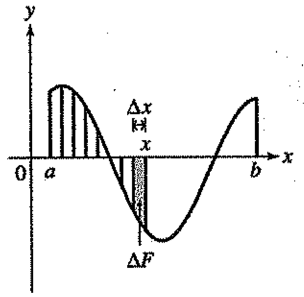
Pictures will vary according to the value of x chosen. (Indeed, this is the point of the exploration.) We show a typical solution here.



1. We have chosen an arbitray x between a and b.
2. We have shaded the region using vertical line segments.
3. The shaded region can be written as $\int_a^x f(t)\,dt$ using the definition of the definite integral in Section 5.2. We use t as a dummy variable because x cannot vary between a and itself.
4. The area of the shaded region is our value of $F(x)$.



5. We have drawn one more vertical shading segment to represent ΔF.
6. We have moved x a distance of Δx so that it rests above the new shading segment.



7. Now the (signed) height of the newly-added vertical segment is $f(x)$.

8. The (signed) area of the segment is $\Delta F = \Delta x \cdot f(x)$, so

$$F'(x) = \lim_{\Delta x \to 0} \frac{\Delta F}{\Delta x} = f(x)$$

Quick Review 5.3

1. $\frac{dy}{dx} = \sin x$

2. $\frac{dy}{dx} = \cos x$

3. $\frac{dy}{dx} = \frac{\sec x \tan x}{\sec x} = \tan x$

4. $\frac{dy}{dx} = \frac{\cos x}{\sin x} = \cot x$

5. $\frac{dy}{dx} = \frac{\sec x \tan x + \sec^2 x}{\sec x + \tan x} = \sec x$

6. $\frac{dy}{dx} = x\left(\frac{1}{x}\right) + \ln x - 1 = \ln x$

7. $\frac{dy}{dx} = \frac{(n+1)x^n}{n+1} = x^n$

8. $\frac{dy}{dx} = -\frac{1}{(2^x+1)^2} \cdot (\ln 2)2^x = -\frac{2^x \ln 2}{(2^x+1)^2}$

9. $\frac{dy}{dx} = xe^x + e^x$

10. $\frac{dy}{dx} = \frac{1}{x^2+1}$

Section 5.3 Exercises

1. (a) $\int_2^2 g(x)\,dx = 0$

(b) $\int_5^1 g(x)\,dx = -\int_1^5 g(x)\,dx = -8$

(c) $\int_1^2 3f(x)\,dx = 3\int_1^2 f(x)\,dx = 3(-4) = -12$

(d)
$$\begin{aligned}\int_2^5 f(x)\,dx &= \int_2^1 f(x)\,dx + \int_1^5 f(x)\,dx\\ &= -\int_1^2 f(x)\,dx + \int_1^5 f(x)\,dx\\ &= 4 + 6 = 10\end{aligned}$$

(e)
$$\begin{aligned}\int_1^5 [f(x) - g(x)]\,dx &= \int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx\\ &= 6 - 8 = -2\end{aligned}$$

(f)
$$\begin{aligned}\int_1^5 [4f(x) - g(x)]\,dx &= \int_1^5 4f(x)\,dx - \int_1^5 g(x)\,dx\\ &= 4\int_1^5 f(x)\,dx - \int_1^5 g(x)\,dx\\ &= 4(6) - 8 = 16\end{aligned}$$

2. (a) $\int_1^9 -2f(x)\,dx = -2\int_1^9 f(x)\,dx = -2(-1) = 2$

(b)
$$\begin{aligned}\int_7^9 [f(x) + h(x)]\,dx &= \int_7^9 f(x)\,dx + \int_7^9 h(x)\,dx\\ &= 5 + 4 = 9\end{aligned}$$

(c)
$$\begin{aligned}\int_7^9 [2f(x) - 3h(x)]\,dx &= \int_7^9 2f(x)\,dx + \int_7^9 3h(x)\,dx\\ &= 2\int_7^9 f(x)\,dx - 3\int_7^9 h(x)\,dx\\ &= 2(5) - 3(4) = -2\end{aligned}$$

(d) $\int_9^1 f(x)\,dx = -\int_1^9 f(x)\,dx = 1$

(e)
$$\begin{aligned}\int_1^7 f(x)\,dx &= \int_1^9 f(x)\,dx + \int_9^7 f(x)\,dx\\ &= \int_1^9 f(x)\,dx - \int_7^9 f(x)\,dx\\ &= -1 - 5 = -6\end{aligned}$$

(f)
$$\begin{aligned}\int_9^7 [h(x) - f(x)]\,dx &= \int_9^7 h(x)\,dx - \int_9^7 f(x)\,dx\\ &= -\int_7^9 h(x)\,dx + \int_7^9 f(x)\,dx\\ &= -4 + 5 = 1\end{aligned}$$

3. (a) $\int_1^2 f(u)\,du = 5$

(b) $\int_1^2 \sqrt{3} f(z)\,dz = \sqrt{3}\int_1^2 f(z)\,dz = 5\sqrt{3}$

(c) $\int_2^1 f(t)\,dt = -\int_1^2 f(t)\,dt = -5$

(d) $\int_1^2 [-f(x)]\,dx = -\int_1^2 f(x)\,dx = -5$

4. (a) $\int_0^{-3} g(t)\,dt = -\int_{-3}^0 g(t)\,dt = -\sqrt{2}$

(b) $\int_{-3}^0 g(u)\,du = \sqrt{2}$

(c) $\int_{-3}^0 [-g(x)]\,dx = -\int_{-3}^0 g(x)\,dx = -\sqrt{2}$

(d) $\int_{-3}^0 \frac{g(r)}{\sqrt{2}}\,dr = \frac{1}{\sqrt{2}}\int_{-3}^0 g(r)\,dr = 1$

5. (a)
$$\begin{aligned}\int_3^4 f(z)\,dz &= \int_3^0 f(z)\,dz + \int_0^4 f(z)\,dz\\ &= \int_0^3 f(z)\,dz + \int_0^4 f(z)\,dz\\ &= -3 + 7 = 4\end{aligned}$$

(b)
$$\begin{aligned}\int_4^3 f(t)\,dt &= \int_4^0 f(t)\,dt + \int_0^3 f(t)\,dt\\ &= -\int_0^4 f(t)\,dt + \int_0^3 f(t)\,dt\\ &= -7 + 3 = -4\end{aligned}$$

6. (a)
$$\begin{aligned}\int_1^3 h(r)\,dr &= \int_1^{-1} h(r)\,dr + \int_{-1}^3 h(r)\,dr\\ &= -\int_{-1}^1 h(r)\,dr + \int_{-1}^3 h(r)\,dr = 6\end{aligned}$$

6. Continued

(b) $-\int_3^1 h(u)\,du = -\int_3^{-1} h(u)\,du - \int_{-1}^1 h(u)\,du$

$= \int_{-1}^3 h(u)\,du - \int_{-1}^1 h(u)\,du = 6$

7. max $\sin(x^2) = \sin(1)$ on $[0, 1]$

$\int_0^1 \sin(x^2)\,dx \le \sin(1) < 1$

8. max $\sqrt{x+8} = 3$ and min $\sqrt{x+8} = 2\sqrt{2}$ on $[0, 1]$

$2\sqrt{2} \le \int_0^1 \sqrt{x+8}\,dx \le 3$

9. $(b-a)\min f(x) \ge 0$ on $[a, b]$

$0 \le (b-a)\min f(x) \le \int_a^b f(x)\,dx$

10. $(b-a)\max f(x) \le 0$ on $[a, b]$

$\int_a^b f(x)dx \le (b-a)\max f(x) \le 0$

11. An antiderivative of x^2-1 is $F(x) = \frac{1}{3}x^3 - x$.

$$av = \frac{1}{\sqrt{3}}\int_0^{\sqrt{3}} (x^2-1)\,dx$$
$$= \frac{1}{\sqrt{3}}\left[F(\sqrt{3}) - F(0)\right]$$
$$= \frac{1}{\sqrt{3}}(0-0) = 0$$

Find $x = c$ in $[0, \sqrt{3}]$ such that $c^2 - 1 = 0$

$c^2 = 1$

$c = \pm 1$

Since 1 is in $[0, \sqrt{3}]$, $x = 1$.

12. An antiderivative of $-\frac{x^2}{2}$ is $F(x) = -\frac{x^3}{6}$.

$$av = \frac{1}{3}\int_0^3\left(-\frac{x^2}{2}\right)dx = \frac{1}{3}[F(3) - F(0)] = \frac{1}{3}\left(-\frac{9}{2}\right) = -\frac{3}{2}$$

Find $x = c$ in $[0, 3]$ such that $-\frac{c^2}{2} = -\frac{3}{2}$.

$c^2 = 3$

$c = \pm\sqrt{3}$

Since $\sqrt{3}$ is in $[0, 3]$, $x = \sqrt{3}$.

13. An antiderivative of $-3x^2 - 1$ is $F(x) = -x^3 - x$.

$$av = \frac{1}{1}\int_0^1 (-3x^2 - 1)\,dx = F(1) - F(0) = -2$$

Find $x = c$ in $[0, 1]$ such that $-3c^2 - 1 = -2$

$-3c^2 = -1$

$c^2 = \frac{1}{3}$

$c = \pm\frac{1}{\sqrt{3}}$

Since $\frac{1}{\sqrt{3}}$ is in $[0, 1]$, $x = \frac{1}{\sqrt{3}}$.

14. An antiderivative of $(x-1)^2$ is $F(x) = \frac{1}{3}(x-1)^3$.

$$av = \frac{1}{3}\int_0^3 (x-1)^2\,dx = \frac{1}{3}[F(3) - F(0)] = \frac{1}{3}\left(\frac{8}{3} + \frac{1}{3}\right) = 1$$

Find $x = c$ in $[0, 3]$ such that $(c-1)^2 = 1$.

$c - 1 = \pm 1$

$c = 2$ or $c = 0$.

Since both are in $[0, 3]$, $x = 0$ or $x = 2$.

15. The region between the graph and the x-axis is a triangle of height 3 and base 6, so the area of the region is $\frac{1}{2}(3)(6) = 9$.

$$av(f) = \frac{1}{6}\int_{-4}^2 f(x)\,dx = \frac{9}{6} = \frac{3}{2}.$$

16. The region between the graph and the x-axis is a rectangle with a half circle of radius 1 cut out. The area of the region is $2(1) - \frac{1}{2}\pi(1)^2 = \frac{4-\pi}{2}$.

$$av(f) = \frac{1}{2}\int_{-1}^1 f(t)\,dt = \frac{1}{2}\left(\frac{4-\pi}{2}\right) = \frac{4-\pi}{4}.$$

17. There are equal areas above and below the x-axis.

$$av(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\,dt = \frac{1}{2\pi}\cdot 0 = 0$$

18. Since $\tan\theta$ is an odd function, there are equal areas above and below the x-axis.

$$av(f) = \frac{1}{\pi/2}\int_{-\pi/4}^{\pi/4} f(\theta)\,d\theta = \frac{2}{\pi}\cdot 0 = 0$$

19. $\int_\pi^{2\pi} \sin x\,dx = -\cos(2\pi) + \cos(\pi)$

$= -2$

20. $\int_0^{\pi/2} \cos x\,dx = \sin\left(\frac{\pi}{2}\right) - \sin(0) = 1$

21. $\int_0^{\pi/1} e^x\,dx = e^1 - e^0 = e - 1$

22. $\int_0^{\pi/4} \sec^2 x\,dx = \tan\left(\frac{\pi}{4}\right) - \tan 0 = 1$

23. $\int_1^4 2x\,dx = x^2\Big|_1^4 = 4^2 - 1^2 = 15$

24. $\int_{-1}^2 3x^2\,dx = x^3\Big|_{-1}^2 = 2^3 - (-1)^3 = 9$

25. $\int_{-2}^6 5\,dx = 5x\Big|_{-2}^6 = 5(6) - 5(-2) = 40$

26. $\int_3^7 8\,dx = 8x\Big|_3^7 = 8(7) - 8(3) = 32$

27. $\int_{-1}^1 \frac{1}{1+x^2}\,dx = \tan^{-1}(1) - \tan^{-1}(-1) = \frac{\pi}{2}$

28. $\int_0^{1/2} \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}\left(\frac{1}{2}\right) - \sin^{-1}(0) = \frac{\pi}{6}$

29. $\int_1^e \frac{1}{x}dx = \ln e - \ln 1 = 1$

30. $\int_1^4 -x^{-2}dx = \frac{1}{x}\Big|_1^4 = \frac{1}{4} - \frac{1}{1} = -\frac{3}{4}$

31. $av(f) = \frac{1}{\pi - 0}\int_0^{\pi} \sin x\,dx$

$$= \frac{1}{\pi}(-\cos\pi - (-\cos 0)) = \frac{2}{\pi}$$

32. $av(f) = \frac{1}{2e - e}\int_e^{2e} \frac{1}{x}dx = \frac{1}{e}(\ln 2e - \ln e)$

$$= \frac{\ln 2}{e}$$

33. $av(f) = \frac{1}{\frac{\pi}{4} - 0}\int_0^{\frac{\pi}{4}} \sec^2 x\,dx = \tan\left(\frac{\pi}{4}\right) - \tan(0)$

$$= \frac{4}{\pi}$$

34. $av(f) = \frac{1}{1-0}\int_0^1 \frac{1}{1+x^2}dx = \tan^{-1}(1) - \tan^{-1}(0)$

$$= \frac{\pi}{4}$$

35. $av(f) = \frac{1}{2-(-1)}\int_{-1}^2 3x^2 + 2x\,dx = \frac{1}{3}\left(x^3 + x^2\right)\Big|_{-1}^2$

$$= 4$$

36. $av(f) = \frac{1}{\frac{\pi}{3} - 0}\int_0^{\frac{\pi}{3}} \sec x \tan x\,dx = \frac{3}{\pi}\left(\sec\left(\frac{\pi}{3}\right) - \sec 0\right)$

$$= \frac{3}{\pi}$$

37. $\min f = \frac{1}{2}$ and $\max f = 1$

$$\frac{1}{2} \le \int_0^1 \frac{1}{1+x^4}dx \le 1$$

38. $f(0.5) = \frac{16}{17}$

$$\left(\frac{1}{2}\right)\left(\frac{16}{17}\right) \le \int_0^{0.5} \frac{1}{1+x^4}\,dx \le \left(\frac{1}{2}\right)(1)$$

$$\frac{8}{17} \le \int_0^{0.5} \frac{1}{1+x^4}\,dx \le \frac{1}{2}$$

$$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) \le \int_{0.5}^1 \frac{1}{1+x^4}\,dx \le \left(\frac{1}{2}\right)\left(\frac{16}{17}\right)$$

$$\frac{1}{4} \le \int_{0.5}^1 \frac{1}{1+x^4}\,dx \le \frac{8}{17}$$

$$\frac{8}{17} + \frac{1}{4} \le \int_0^1 \frac{1}{1+x^4}\,dx \le \frac{1}{2} + \frac{8}{17}$$

$$\frac{49}{68} \le \int_0^1 \frac{1}{1+x^4}\,dx \le \frac{33}{34}$$

39. Yes, $\int_a^b av(f)\,dx = \int_a^b f(x)\,dx$.

This is because $av(f)$ is a constant, so

$$\int_a^b av(f)\,dx = \left[av(f)\cdot x\right]_a^b$$
$$= av(f)\cdot b - av(f)\cdot a$$
$$= (b-a)av(f)$$
$$= (b-a)\left[\frac{1}{b-a}\int_a^b f(x)\,dx\right]$$
$$= \int_a^b f(x)\,dx$$

40. **(a)** 300 mi

(b) $\frac{150\text{ mi}}{30\text{ mph}} + \frac{150\text{ mi}}{50\text{ mph}} = 8\text{ h}$

(c) $\frac{300\text{ mi}}{8\text{ h}} = 37.5\text{ mph}$

(d) The average speed is the total distance divided by the total time. Algebraically, $\frac{d_1 + d_2}{t_1 + t_2}$. The driver computed $\frac{1}{2}\left(\frac{d_1}{t_1} + \frac{d_2}{t_2}\right)$. The two expressions are not equal.

41. Time for first release $= \frac{1000\text{ m}^3}{10\text{ m}^3/\text{min}} = 100\text{ min}$

Time for second release $= \frac{100\text{ m}^3}{20\text{ m}^3/\text{min}} = 50\text{ min}$

Average rate $= \frac{\text{total released}}{\text{total time}} = \frac{2000\text{ m}^3}{150\text{ min}} = 13\frac{1}{3}\text{ m}^3/\text{min}$

42. $\int_0^1 \sin x\,dx \le \int_0^1 x\,dx = \left[\frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$

43. $\int_0^1 \sec x\,dx \ge \int_0^1\left(1 + \frac{x^2}{2}\right)dx = \left[x + \frac{x^3}{6}\right]_0^1 = \frac{7}{6}$

44. Let $L(x) = cx + d$. Then the average value of f on $[a, b]$ is

$$av(f) = \frac{1}{b-a}\int_a^b (cx+d)dx$$
$$= \frac{1}{b-a}\left[\left(\frac{cb^2}{2} + db\right) - \left(\frac{ca^2}{2} + da\right)\right]$$
$$= \frac{1}{b-a}\left[\frac{c(b^2 - a^2)}{2} + d(b-a)\right]$$
$$= \frac{c(b+a) + 2d}{2}$$
$$= \frac{(ca+d) + (cb+d)}{2}$$
$$= \frac{L(a) + L(b)}{2}$$

45. False. For example, $\sin 0 = \sin\pi = 0$, but the average value of $\sin x$ on $[0, \pi]$ is greater than 0.

46. False. For example, $\int_{-3}^3 2x\,dx = 0$ but $2(-3) \ne 2(3)$

47. A. There is no rule for the multiplication of functions.

48. D. There is no rule for the negation of the bounds.

49. B. $av(f) = \frac{1}{5-1}\int_1^5 \cos x dx = \frac{1}{4}(\sin 5 - \sin 1)$
$= -0.450.$

50. C. $10 = \frac{1}{b-a}\int_a^b F(x)dx$

$10(b-a) = \int_a^b f(x)\,dx$

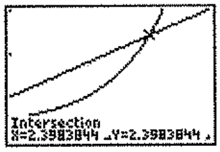
51. **(a)** Area $= \frac{1}{2}$bh

(b) $\frac{h}{2b}x^2 + C$

(c) $\int_0^b y(x)\,dx = \left[\frac{h}{2b}x^2\right]_0^b = \frac{hb^2}{2b} = \frac{1}{2}bh$

52. $av(x^k) = \frac{1}{k}\int_0^k x^k dx = \frac{1}{k}\left[\frac{1}{k+1}x^{k+1}\right]_0^k = \frac{k^{k+1}}{k(k+1)}$

Graph $y_1 = \frac{x^{x+1}}{x(x+1)}$ and $y_2 = x$ on a graphing calculator and find the point of intersection for $x > 1$.



[1, 3] by [0, 3]
Thus, $k \approx 2.39838$

53. An antiderivative of $F'(x)$ is $F(x)$ and an antiderivative of $G'(x)$ is $G(x)$.

$\int_a^b F(x)\,dx = F(b) - F(a)$

$\int_a^b G'(x)\,dx = G(b) - G(a)$

Since $F'(x) = G'(x)$, $\int_a^b F'(x)\,dx = \int_a^b G'(x)\,dx$, so
$F(b) - F(a) = G(b) - G(a)$.

Quick Quiz Sections 5.1–5.3

1. D. $\int_a^b (F(x)+3)\,dx = a + 2b + \int_a^b 3dx$
$a + 2b + 3b - 3a = 5b - 2a$

2. B.

3. C. $\int_2^2 x^2 dx = \left.\frac{x^3}{3}\right|_2^2 = \frac{2^3}{3} - \frac{2^3}{3} = 0.$

4. **(a)** $f''(x) = 6x + 12$

$\int f''(x)dx = 3x^2 + 12x + c$
$y = 4x - 5$
$m = f' = 4$
$3(0)^2 + 12(0) + c = 4$
$c = 4$
$\int f'(x)dx = \int (3x^2 + 12x + 4)dx$
$f(x) = x^3 + 6x^2 + 4x + c$
$f(0) = (0)^3 + 6(0)^2 + 4(0) + c = -5$
$c = -5$
$f(x) = x^3 + 6x^2 + 4x - 5$

(b) $av(f) = \frac{1}{1-(-1)}\int_{-1}^1 (x^3 + 6x^2 + 4x - 5)dx$

$$\frac{1}{2}\left(\frac{x^4}{4} + 2x^3 + 2x^2 - 5x\right)\Bigg|_{-1}^{1} = -3$$

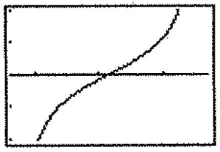
Section 5.4 Fundamental Theorem of Calculus (pp. 294–305)

Exploration 1 Graphing NINT *f*

2. The function $y = \tan x$ has vertical asymptotes at all odd multiples of $\frac{\pi}{2}$. There are six of these between –10 and 10.

3. In attempting to find $F(-10) = \int_3^{-10} \tan(t)\,dt + 5$, the calculator must find a limit of Riemann sums for the integral, using values of tan t for t between – 10 and 3. The large positive and negative values of tan t found near the asymptotes cause the sums to fluctuate erratically so that no limit is approached. (We will see in Section 8.3 that the "areas" near the asymptotes are infinite, although NINT is not designed to determine this.)

4. $y = \tan x$

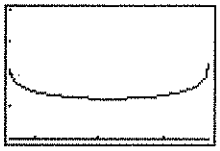


[1.6, 4.7] by [–2, 2]

5. The domain of this continuous function is the open interval $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

6. The domain of F is the same as the domain of the continuous function in step 4, namely $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$.

7. We need to choose a closed window narrower than $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ to avoid the asymptotes.

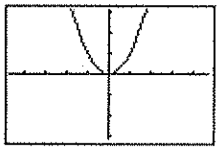


[1.6, 4.7] by [0, 16]

8. The graph of F looks the graph in step 7. It would be decreasing on $\left(\frac{\pi}{2}, \pi\right]$ and increasing on $\left[\pi, \frac{3\pi}{2}\right)$, with vertical asymptotes at $x = \frac{\pi}{2}$ and $x = \frac{3\pi}{2}$.

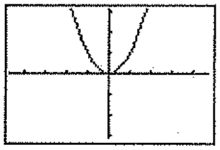
Exploration 2 The Effect of Changing a in $\int_a^x f(t)\,dt$

1.



[−4.7, 4.7] by [−3.1, 3.1]

2.



[−4.7, 4.7] by [−3.1, 3.1]

3. Since NINT $(x^2, x, 0, 0) = 0$, the x-intercept is 0.

4. Since NINT $(x^2, x, 5, 5) = 0$, the x-intercept is 5.

5. Changing a has no effect on the graph of $y = \frac{d}{dx}\int_a^x f(t)\,dt$. It will always be the same as the graph of $y = f(x)$.

6. Changing a shifts the graph of $y = \int_a^x f(t)\,dt$ vertically in such a way that a is always the x-intercept. If we change from a_1 to a_2, the distance of the vertical shift is $\int_{a_2}^{a_1} f(t)\,dt$.

Quick Review 5.4

1. $\frac{dy}{dx} = \cos(x^2) \cdot 2x = 2x\cos(x^2)$

2. $\frac{dy}{dx} = 2(\sin x)(\cos x) = 2\sin x\cos x$

3. $\frac{dy}{dx} = 2(\sec x)(\sec x\tan x) - 2(\tan x)(\sec^2 x)$

$= 2\sec^2 x\tan x - 2\tan x\sec^2 x = 0$

4. $\frac{dy}{dx} = \frac{3}{3x} - \frac{7}{7x} = 0$

5. $\frac{dy}{dx} = 2^x \ln 2$

6. $\frac{dy}{dx} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$

7. $\frac{dy}{dx} = \frac{(-\sin x)(x) - (\cos x)(1)}{x^2} = -\frac{x\sin x + \cos x}{x^2}$

8. $\frac{dy}{dt} = \cos t, \frac{dy}{dt} = -\sin t$

$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\cos t}{-\sin t} = -\cot t$

9. Implicitly differentiate:

$x\frac{dy}{dx} + (1)y + 1 = 2y\frac{dy}{dx}$

$\frac{dy}{dx}(x - 2y) = -(y+1)$

$\frac{dy}{dx} = -\frac{y+1}{x-2y} = \frac{y+1}{2y-x}$

10. $\frac{dy}{dx} = \frac{1}{dx/dy} = \frac{1}{3x}$

Section 5.4 Exercises

1. $\frac{dy}{dx} = \frac{d}{dx}\int_0^x \left(\sin^2 t\right)dt = \sin^2 x$

2. $\frac{dy}{dx} = \frac{d}{dx}\int_2^x \left(3t + \cos t^2\right)dt = 3x + \cos x^2$

3. $\frac{dy}{dx} = \frac{d}{dx}\int_0^x \left(t^3 - t\right)^5 dt = \left(x^3 - x\right)^5$

4. $\frac{dy}{dx} = \frac{d}{dx}\int_{-2}^x \sqrt{1 + e^{5t}}\,dt = \sqrt{1 + e^{5x}}$

5. $\frac{dy}{dx} = \frac{d}{dx}\int_0^x \left(\tan^3 u\right)dt = \tan^3 x$

6. $\frac{dy}{dx} = \frac{d}{dx}\int_4^x e^u \sec u\,du = e^x \sec x$

7. $\frac{dy}{dx} = \frac{d}{dx}\int_7^x \frac{1+t}{1+t^2}dt = \frac{1+x}{1+x^2}$

8. $\frac{dy}{dx} = \frac{d}{dx}\int_-^x \frac{2-\sin t}{3+\cos t}dt = \frac{2-\sin x}{3+\cos x}$

9. $\frac{dy}{dx} = \frac{d}{dx}\int_0^{x^2} e^{t^3}dt = e^{x^2}\frac{du}{dx} = 2x\,e^{x^2}$

10. $\frac{dy}{dx} = \frac{d}{dx}\int_6^{x^2} \cot 3\,t\,dt = \cot x^2 \frac{du}{dx} = 2x\cot 3x^2$

11. $\frac{dy}{dx} = \frac{d}{dx}\int_2^{5x} \frac{\sqrt{1+u^2}}{u}du = \frac{\sqrt{1+25x^2}}{x}$

12. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int^{-x}\dfrac{1+\sin^2 u}{1+\cos^2 u}du=\dfrac{1+\sin^2(\ -x)}{1+\cos^2(\ -x)}$

13. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int_x^6 \ln\left(1+t^2\right)dt=-\dfrac{d}{dx}\int_6^x \ln\left(1+t^2\right)dt$

$=\ln\left(1+x^2\right)$

14. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int_x^7 \sqrt{2t^4+t+1}\ dt=-\dfrac{d}{dx}\int_7^x \sqrt{2t^4+t+1}\ dt$

$=-\sqrt{2x^4+x+1}$

15. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int_{x^3}^5 \dfrac{\cos t}{t^2-2}dt=-\dfrac{d}{dx}\int_5^{x^3}\dfrac{\cos t}{t^2-2}dt=\dfrac{\cos x^3}{x^6-2}\dfrac{du}{dx}$

$=-\dfrac{3x^2\cos x^3}{x^6+2}$

16. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int_{5x^2}^{25}\dfrac{t^2-2t+9}{t^3+6}dt=-\dfrac{d}{dx}\int_{25}^{5x^2}\dfrac{t\ -2t+9}{t^3+6}dt$

$=\dfrac{25\ x^4-10\ x^2+9}{125\ x^6+6}\dfrac{du}{dx}=\dfrac{250\ x^5-100\ x^3+90x}{125\ x^6+6}$

17. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int_{\sqrt{x}}^0 \sin\left(r^2\right)dr=-\dfrac{d}{dx}\int_0^{\sqrt{x}}\sin\left(r^2\right)dr$

$=-\sin x\dfrac{du}{dx}=-\dfrac{\sin x}{2\sqrt{x}}$

18. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int_{3x^2}^{10}\ln\left(2+p^2\right)dp=-\dfrac{d}{dx}\int_{10}^{3x^2}\ln\left(2+p^2\right)dp$

$=-\ln\left(2+p^2\right)\dfrac{du}{dx}=-6x\ln\left(2+9x^4\right)$

19. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int_{x^2}^{x^3}\cos\ 2t\ dt=\cos 2x^3\dfrac{du}{dx}+\cos 2x^2\dfrac{du}{dx}$

$=3x^2\cos 2x^3+2x\cos 2x^2$

20. $\dfrac{dy}{dx}=\dfrac{d}{dx}\int_{\sin x}^{\cos x}t^2 dt=\cos^2 x\dfrac{du}{dx}-\sin^2 x\dfrac{du}{dx}$

$=-\sin x\ \cos^2 x-\cos x\ \sin^2 x$

21. $y=\int_5^x \sin^3 t\ dt$

22. $y=\int_8^x e^{-t}\tan t\ dt$

23. $\left|E_{S_{10n}}\right|=10^{-4}\left|E_{S_n}\right|$

24. $y=\int_{-3}^x \sqrt{3-\cos t}\ dt+4$

25. $y=\int_7^x \cos^2 5t\ dt-2$

26. $y=\int_0^x e^{\sqrt{t}}dt+1$

27. $\int_{1/2}^3\left(2-\dfrac{1}{x}\right)dx=\left[2x-\ln|x|\right]_{1/2}^3$

$=(6-\ln 3)-\left(1-\ln\dfrac{1}{2}\right)$

$=5-\ln 3+\ln\dfrac{1}{2}$

$=5-\ln 3-\ln 2$

$=5-\ln 6\approx 3.208$

28. $\int_2^{-1}3^x dx=\left[\left(\dfrac{1}{\ln 3}\right)3^x\right]_2^{-1}$

$=\left(\dfrac{1}{\ln 3}\right)\left(\dfrac{1}{3}-9\right)$

$=-\dfrac{26}{3\ln 3}\approx -7.889$

29. $\int_0^1 (x^2+\sqrt{x})\,dx=\left[\dfrac{1}{3}x^3+\dfrac{2}{3}x^{3/2}\right]_0^1=\left(\dfrac{1}{3}+\dfrac{2}{3}\right)-(0+0)=1$

30. $\int_0^5 x^{3/2}\ dx=\left[\dfrac{2}{5}x^{5/2}\right]_0^5=\dfrac{2}{5}(25\sqrt{5}-0)=10\sqrt{5}\approx 22.361$

31. $\int_1^{32}x^{-6/5}dx=\left[-5x^{-1/5}\right]_1^{32}=-5\left(\dfrac{1}{2}-1\right)=\dfrac{5}{2}$

32. $\int_{-2}^{-1}\dfrac{2}{x^2}dx=2\int_{-2}^{-1}x^{-2}dx=2\left[-x^{-1}\right]_{-2}^{-1}=2\left[1-\dfrac{1}{2}\right]=1$

33. $\int_0^\pi \sin x\,dx=\left[-\cos x\right]_0^\pi=1-(-1)=2$

34. $\int_0^\pi (1+\cos x)\,dx=\left[x+\sin x\right]_0^\pi$

$=(\pi+0)-(0+0)$

$=\pi\approx 3.142$

35. $\int_0^{\pi/3}2\sec^2\theta\ d\theta=2\left[\tan\theta\right]_0^{\pi/3}$

$=2(\sqrt{3}-0)$

$=2\sqrt{3}\approx 3.464$

36. $\int_{\pi/6}^{5\pi/6}\csc^2\theta\ d\theta=\left[-\cot\theta\right]_{\pi/6}^{5\pi/6}$

$=\sqrt{3}-(-\sqrt{3})$

$=2\sqrt{3}\approx 3.464$

37. $\int_{\pi/4}^{3\pi/4}\csc x\cot x\ dx=\left[-\csc x\right]_{\pi/4}^{3\pi/4}=(-\sqrt{2})-(-\sqrt{2})=0$

38. $\int_0^{\pi/3}4\sec x\tan x\ dx=4\left[\sec x\right]_0^{\pi/3}=4(2-1)=4$

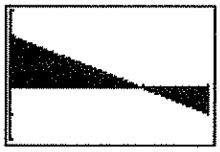
39. $\int_{-1}^1 \left(r+1\right)^2 dr=\left[\dfrac{1}{3}\left(r+1\right)^3\right]_{-1}^1=\dfrac{8}{3}-0=\dfrac{8}{3}$

40. $\int_0^4 \frac{1-\sqrt{u}}{\sqrt{u}}\,du = \int_0^4\left(u^{-1/2}-1\right)du$

$= \left[2u^{-1/2} - u\right]_0^4$

$= (4-4)-(0-0) = 0$

41. Graph $y = 2 - x$.



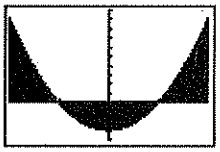
[0, 3] by [–2, 3]

Over [0, 2]: $\int_0^2 (2-x)\,dx = \left[2x - \frac{1}{2}x^2\right]_0^2 = 2$

Over [2, 3]: $\int_2^3 (2-x)\,dx = \left[2x - \frac{1}{2}x^2\right]_2^3 = \frac{3}{2} - 2 = -\frac{1}{2}$

Total area $= |2| + \left|-\frac{1}{2}\right| = \frac{5}{2}$

42. Graph $y = 3x^2 - 3$.



[–2, 2] by [–4, 10]

Over [–2, –1]:

$\int_{-2}^{-1}\left(3x^2-3\right)dx = \left[x^3 - 3x\right]_{-2}^{-1} = 2 - (-2) = 4$

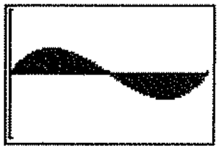
Over [–1, 1]:

$\int_{-1}^{1}\left(3x^2-3\right)dx = \left[x^3 - 3x\right]_{-1}^{1} = -2 - 2 = -4$

Over [1, 2]: $\int_1^2\left(3x^2-3\right)dx = \left[x^3-3x\right]_1^2 = 2 - (-2) = 4$

Total area $= |4| + |-4| + |4| = 12$

43. Graph $y = x^3 - 3x^2 - 2x$.



[0, 2] by [–1, 1]

Over [0, 1]:

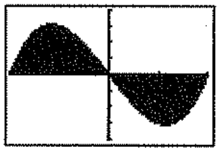
$\int_0^1\left(x^3 - 3x^2 + 2x\right)dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_0^1 = \frac{1}{4} - 0 = \frac{1}{4}$

Over [1, 2]:

$\int_1^2\left(x^3 - 3x^2 + 2x\right)dx = \left[\frac{1}{4}x^4 - x^3 + x^2\right]_1^2 = 0 - \frac{1}{4} = -\frac{1}{4}$

Total area $= \left|\frac{1}{4}\right| + \left|-\frac{1}{4}\right| = \frac{1}{2}$

44. Graph $y = x^3 - 4x$.



[–2, 2] by [–4, 4]

Over [–2, 0]:

$\int_{-2}^{0}\left(x^3 - 4x\right)dx = \left[\frac{1}{4}x^4 - 2x^2\right]_{-2}^{0} = 0 - (-4) = 4$

Over [0, 2]:

$\int_0^2\left(x^3 - 4x\right)dx = \left[\frac{1}{4}x^4 - 2x^2\right]_0^2 = -4 - 0 = -4$

Total area $= |4| + |-4| = 8$

45. First, find the area under the graph of $y = x^2$.

$\int_0^1 x^2\,dx = \left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{3}$

Next find the area under the graph of $y = 2 - x$.

$\int_1^2 (2-x)\,dx = \left[2x - \frac{1}{2}x^2\right]_1^2 = 2 - \frac{3}{2} = \frac{1}{2}$

Area of the shaded region $= \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

46. First find the area under the graph of $y = \sqrt{x}$.

$\int_0^1 x^{1/2}\,dx = \left[\frac{2}{3}x^{3/2}\right]_0^1 = \frac{2}{3}$

Next find the area under the graph of $y = x^2$.

$\int_1^2 x^2\,dx = \left[\frac{1}{3}x^3\right]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}$

Area of the shaded region $= \frac{2}{3} + \frac{7}{3} = 3$

47. First, find the area under the graph of $y = 1 + \cos x$.

$\int_0^{\pi}(1 + \cos x)\,dx = \left[x + \sin x\right]_0^{\pi} = \pi$

The area of the rectangle is 2π.

Area of the shaded region $= 2\pi - \pi = \pi$.

48. First, find the area of the region between $y = \sin x$ and the x-axis for $\left[\frac{\pi}{6}, \frac{5\pi}{6}\right]$.

$\int_{\pi/6}^{5\pi/6} \sin x\,dx = \left[-\cos x\right]_{\pi/6}^{5\pi/6} = \frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right) = \sqrt{3}$

The area of the rectangle is $\left(\sin\frac{\pi}{6}\right)\left(\frac{2\pi}{3}\right) = \frac{\pi}{3}$

Area of the shaded region $= \sqrt{3} - \frac{\pi}{3}$

49. $\text{NINT}\left(\frac{1}{3 + 2\sin x}, x, 0, 10\right) \approx 3.802$

50. $\text{NINT}\left(\frac{2x^4-1}{x^4-1},\ x,\ -0.8,\ 0.8\right) \approx 1.427$

51. $\frac{1}{2}\text{NINT}\left(\sqrt{\cos x},\ x, -1, 1\right) \approx 0.914$

52. $\sqrt{8-2x^2} \ge 0$ between $x = -2$ and $x = 2$

$\text{NINT}(\sqrt{8-2x^2}, x, -2, 2) \approx 8.886$

53. Plot $y_1 = \text{NINT}\left(e^{-t^2},\ t,\ 0,\ x\right), y_2 = 0.6$ in a [0, 1] by [0, 1] window, then use the intersect function to find $x \approx 0.699$.

54. When $y = 0$, $x = 1$.

$y^3 = 1 - x^3$

$y = \sqrt[3]{1-x^3}$

$\text{NINT}(\sqrt[3]{1-x^3},\ x,\ 0,\ 1) \approx 0.883$

55. $\int_a^x f(t)\,dt + K = \int_b^x f(t)\,dt$

$$K = -\int_a^x f(t)\,dt + \int_b^x f(t)\,dt$$
$$= \int_x^a f(t)\,dt + \int_b^x f(t)\,dt$$
$$= \int_b^x f(t)\,dt$$
$$K = \int_2^{-1} (t^2 - 3t + 1)\,dt$$
$$= \left[\frac{1}{3}t^3 - \frac{3}{2}t^2 + t\right]_2^{-1}$$
$$= \left[-\frac{1}{3} - \frac{3}{2} + (-1)\right] - \left[\frac{8}{3} - 6 + 2\right] = -\frac{3}{2}$$

56. To find an antiderivative of $\sin^2 x$, recall from trigonometry that $\cos 2x = 1 - 2\sin^2 x$, so $\sin^2 x = \frac{1}{2} - \frac{1}{2}\cos 2x$.

$$K = \int_2^0 \sin^2 t\,dt$$
$$= \int_2^0 \left[\frac{1}{2} - \frac{1}{2}\cos(2x)\right] dx$$
$$= \left[\frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_2^0$$
$$= \left[\frac{1}{2}x - \frac{1}{2}\sin x \cos x\right]_2^0$$
$$= 0 - \left(1 - \frac{\sin 2\cos 2}{2}\right) = \frac{\sin 2\cos 2 - 2}{2} \approx -1.189$$

57. (a) $H(0) = \int_0^0 f(t)\,dt = 0$

(b) $H'(x) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = f(x)$

$H'(x) > 0$ when $f(x) > 0$.

H is increasing on $[0, 6]$.

(c) H is concave up on the open interval where $H''(x) = f'(x) > 0$.

$f'(x) > 0$ when $9 < x \le 12$.

H is concave up on $(9, 12)$.

(d) $H(12) = \int_0^{12} f(t)\,dt > 0$ because there is more area above the x-axis than below the x-axis.

$H(12)$ is positive.

(e) $H'(x) = f(x) = 0$ at $x = 6$ and $x = 12$. Since $H'(x) = f(x) > 0$ on $[0, 6)$, the values of H are increasing to the left of $x = 6$, and since $H'(x) = f(x) < 0$ on $(6, 12]$, the values of H are decreasing to the right of $x = 6$. H achieves its maximum value at $x = 6$.

(f) $H(x) > 0$ on $(0, 12]$. Since $H(0) = 0$, H achieves its minimum value at $x = 0$.

58. (a) $s'(t) = f(t)$. The velocity at $t = 5$ is $f(5) = 2$ units/sec.

(b) $s''(t) = f'(t) < 0$ at $t = 5$ since the graph is decreasing, so acceleration at $t = 5$ is negative.

(c) $s(3) = \int_0^3 f(x)\,dx = \frac{1}{2}(3)(3) = 4.5$ units

(d) s has its largest value at $t = 6$ sec since $s'(6) = f(6) = 0$ and $s''(6) = f'(6) < 0$.

(e) The acceleration is zero when $s''(t) = f'(t) = 0$. This occurs when $t = 4$ sec and $t = 7$sec.

(f) Since $s(0) = 0$ and $s'(t) = f(t) > 0$ on $(0, 6)$, the particle moves away from the origin in the positive direction on $(0,6)$. The particle then moves in the negative direction, towards the origin, on $(6, 9)$ since $s'(t) = f(t) < 0$ on $(6, 9)$ and the area below the x-axis is smaller than the area above the x-axis.

(g) The particle is on the positive side since

$s(9) = \int_0^9 f(x)\,dx > 0$ (the area below the x-axis is smaller than the area above the x-axis).

59. (a) $s'(3) = f(3) = 0$ units/sec

(b) $s''(3) = f'(3) > 0$ so acceleration is positive.

(c) $s(3) = \int_0^3 f(x)\,dx = \frac{1}{2}(-6)(3) = -9$ units

(d) $s(6) = \int_0^6 f(x)\,dx = \frac{1}{2}(-6)(3) + \frac{1}{2}(6)(3) = 0$, so the particle passes through the origin at $t = 6$ sec.

(e) $s''(t) = f'(t) = 0$ at $t = 7$ sec

(f) The particle is moving away from the origin in the negative direction on $(0,3)$ since $s(0) = 0$ and $s'(t) < 0$ on $(0, 3)$. The particle is moving toward the origin on $(3, 6)$ since $s'(t) > 0$ on $(3, 6)$ and $s(6) = 0$. The particle moves away from the origin in the positive direction for $t > 6$ since $s'(t) > 0$.