25. Set $c'(x) = \frac{c(x)}{x}$: $3x^2 - 20x + 30 = x^2 - 10x + 30$. The only positive solution is $x = 5$, so average cost is minimized at a production level of 5000 units. Note that

$\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = 2 > 0$ for all positive x, so the Second Derivative Test Confirms the minimum.

26. Set $c'(x) = c(x)/x: xe^x + e^x - 4x = e^x - 2x$. The only positive solution is $x = \ln 2$, so average cost is minimized at a production level of 1000 ln 2, which is about 693 units. Note that $\frac{d^2}{dx^2}\left(\frac{c(x)}{x}\right) = e^x > 0$ for all positive x, so the Second Derivative Test confirms the minimum.

27. Revenue: $r(x) = [200 - 2(x-50)]x = -2x^2 + 300x$

Cost: $c(x) = 6000 + 32x$

Profit: $p(x) = r(x) - c(x)$

$= -2x^2 + 268x - 6000, 50 \le x \le 80$

Since $p'(x) = -4x + 268 = -4(x-67)$, the critical point occurs at $x = 67$. This value represents the maximum because $p''(x) = -4$, which is negative for all x in the domain. The maximum profit occurs if 67 people go on the tour.

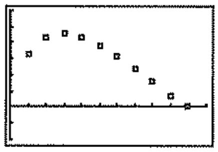
28. (a) $f'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x}(1-x)$

The critical point occurs at $x = 1$. Since $f'(x) > 0$ for $0 \le x < 1$ and $f'(x) < 0$ for $x > 1$, the critical point corresponds to the maximum value of f. The absolute maximum of f occurs at $x = 1$.

(b) To find the values of b, use grapher techniques to solve $xe^{-x} = 0.1e^{-0.1}$, $xe^{-x} = 0.2e^{-0.2}$, and so on. To find the values of A, calculate $(b-a)\,ae^{-2}$, using the unrounded values of b. (Use the *list* features of the grapher in order to keep track of the unrounded values for part (d).)

a	b	A
0.1	3.71	0.33
0.2	2.86	0.44
0.3	2.36	0.46
0.4	2.02	0.43
0.5	1.76	0.38
0.6	1.55	0.31
0.7	1.38	0.23
0.8	1.23	0.15
0.9	1.11	0.08
1.0	1.00	0.00

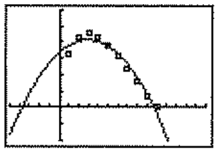
(c)



[0, 1.1] by [−0.2, 0.6]

(d) Quadratic:

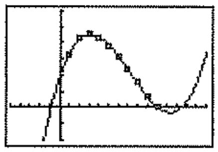
$A \approx -0.91a^2 + 0.54a + 0.34$



[−0.5, 1.5] by [−0.2, 0.6]

Cubic:

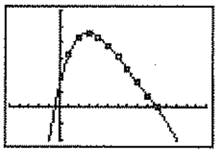
$A \approx 1.74a^3 - 3.78a^2 + 1.86a + 0.19$



[−0.5, 1.5] by [−0.2, 0.6]

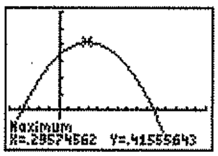
Quartic:

$A \approx -1.92a^4 + 5.96a^3 - 6.87a^2 + 2.71a + 0.12$



[−0.5, 1.5] by [−0.2, 0.6]

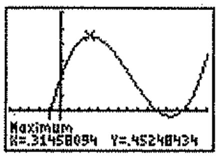
(e) Quadratic:



[−0.5, 1.5] by [−0.2, 0.6]

According to the quadratic regression equation, the maximum area occurs at $a \approx 0.30$ and is approximately 0.42.

Cubic:

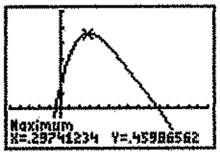


[−0.5, 1.5] by [−0.2, 0.6]

According to the cubic regression equation, the maxiumu area occurs at $a \approx 0.31$ and is approximately 0.45.

28. Continued

Quartic:



$[-0.5, 1.5]$ by $[-0.2, 0.6]$

According to the quartic regression equation the maximum area occurs at $a \approx 0.30$ and is approximately 0.46.

29. (a) $f'(x)$ is a quadratic polynominal, and as such it can have 0, 1, or 2 zeros. If it has 0 or 1 zeros, then its sign never changes, so $f(x)$ has no local extrema.
If $f'(x)$ has 2 zeros, then its sign changes twice, and $f(x)$ has 2 local extrema at those points.

(b) Possible answers:
No local extrema: $y = x^3$;
2 local extrema: $y = x^3 - 3x$

30. Let x be the length in inches of each edge of the square end, and let y be the length of the box. Then we require $4x + y \le 108$. Since our goal is to maximize volume, we assume $4x + y = 108$ and so $y = 108 - 4x$. The volume is $V(x) = x^2(108 - 4x) = 108x^2 - 4x^3$, where $0 < x < 27$. Then $V' = 216x - 12x^2 = -12x(x - 18)$, so the critical point occurs at $x = 18$ in. Since $V'(x) > 0$ for $0 < x < 18$ and $V'(x) < 0$ for $18 < x < 27$, the critical point corresponds to the maximum volume. The dimensions of the box with the largest possible volume are 18 in. by 18 in. by 36 in.

31. Since $2x + 2y = 36$, we know that $y = 18 - x$. In part (a), the radius is $\frac{x}{2\pi}$ and the height is $18 - x$, and so the volume is given by

$$\pi r^2 h = \pi\left(\frac{x}{2\pi}\right)^2 (18 - x) = \frac{1}{4\pi} x^2(18 - x).$$

In part (b), the radius is and the height is $18 - x$, and so the volume is given by $\pi r^2 h = \pi x^2(18 - x)$. Thus, each problem requires us to find the value of x that maximizes $f(x) = x^2(18 - x)$ in the interval $0 < x < 18$, so the two problems have the same answer.
To solve either problem, note that $f(x) = 18^2 - x^3$ and so $f'(x) = 36x - 3x^2 = -3x(x - 12)$. The critical point occursat $x = 12$. Since $f'(x) > 0$ for $0 < x < 12$ and $f'(x) < 0$ for $12 < x < 18$, the critical point corresponds to the maximum value of $f(x)$. To maximize the volume in either part (a) or (b), let $x = 12$ cm and $y = 6$ cm.

32. Note that $h^2 + r^2 = 3$ and so $r = \sqrt{3 - h^2}$. Then the volume is given by $V = \frac{\pi}{3} r^2 h = \frac{\pi}{3}(3 - h^2)h = \pi h - \frac{\pi}{3} h^3$ for $0 < h < \sqrt{3}$, and so $\frac{dV}{dh} = \pi - \pi h^2 = \pi(1 - h^2)$. The critical point (for $h > 0$) occurs at $h = 1$. Since $\frac{dV}{dh} > 0$ for $0 < h < 1$ and $\frac{dV}{dh} < 0$ for $1 < h < \sqrt{3}$, the critical point corresponds to the maximum volume. The cone of greatest volume has radius $\sqrt{2}$ m, height 1 m, and volume $\frac{2\pi}{3}\text{m}^3$.

33. (a) We require $f(x)$ to have a critical point at $x = 2$. Since $f'(x) = 2x - ax^{-2}$, we have $f'(2) = 4 - \frac{a}{4}$ and so our requirement is that $4 - \frac{a}{4} = 0$. Therefore, $a = 16$. To verify that the critical point corresponds to a local minimum, note that we now have $f'(x) = 2x - 16x^{-2}$ and so $f''(x) = 2 + 32x^{-3}$, so $f''(2) = 6$, which is positive as expected. So, use $a = -16$.

(b) We require $f''(1) = 0$. Since $f'' = 2 + 2ax^{-3}$, we have $f''(1) = 2 + 2a$, so our requirement is that $2 + 2a = 0$. Therefore, $a = -1$. To verify that $x = 1$ is in fact an inflection point, note that we now have $f''(x) = 2 - 2x^{-3}$, which is negative for $0 < x < 1$ and positive for $x > 1$. Therefore, the graph of f is concave down in the interval $(0, 1)$ and concave up in the interval $(1, \infty)$, So, use $a = -1$.

34. $f'(x) = 2x - ax^{-2} = \frac{2x^3 - a}{x^2}$, so the only sign change in $f'(x)$ occurs at $x = \left(\frac{a}{2}\right)^{1/3}$, where the sign changes from negative to positive. This means there is a local minimum at that point, and there are local maxima.

35. (a) Note that $f'(x) = 3x^2 + 2ax + b$. We require $f'(-1) = 0$ and $f'(3) = 0$, which give $3 - 2a + b = 0$ and $27 + 6a + b = 0$. Substracting the first equation from the second, we have $24 + 8a = 0$ and so $a = -3$. Substituting into the first equation, we have $9 + b = 0$, so $b = -9$. Therefore, our equation for $f(x)$ is $f(x) = x^3 - 3x^2 - 9x$. To verify that we have a local maximum at $x = -1$ and a local minimum at $x = 3$, note that $f'(x) = 3x^2 - 6x - 9 = 3(x + 1)(x - 3)$, which is positive for $x < -1$, negative for $-1 < x < 3$, and positive for $x > 3$. So, use $a = -3$ and $b = -9$.

(b) Note that $f'(x) = 3x^2 + 2ax + b$ and $f''(x) = 6x + 2a$. We require $f'(4) = 0$ and $f''(1) = 0$, which give $48 + 8a + b = 0$ and $6 + 2a = 0$. By the second equation, $a = -3$, and so the first equation becomes $48 - 24 + b = 0$. Thus $b = -24$. To verify that we have a local minimum at $x = 4$, and an inflection point *at* $x = 1$, note that we now have $f''(x) = 6x - 6$. Since f'' changes sign at $x = 1$ and is positive at $x = 4$, the desired conditions are satisfied. So, use $a = -3$ and $b = -24$.

36. Refer to the illustration in the problem statement. Since $x^2+y^2=9$, we have $x=\sqrt{9-y^2}$. Then the volume of the cone is given by

$$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi x^2(y+3)$$
$$=\frac{1}{3}\pi(9-y^2)(y+3)$$
$$=\frac{\pi}{3}(-y^3-3y^2+9y+27),$$

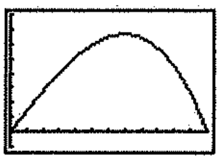
for $-3<y<3$.

Thus $\frac{dV}{dy}=\frac{\pi}{3}(-3y^2-6y+9)=-\pi(y^2+2y-3)$

$=-\pi(y+3)(y-1)$, so the critical point in the interval $(-3, 3)$ is $y=1$. Since $\frac{dV}{dy}>0$ for $-3<y<1$ and $\frac{dV}{dy}<0$ for $1<y<3$, the critical point does correspond to the maximum value, which is $V(1)=\frac{32\pi}{3}$ cubic units.

37. (a) Note that $w^2+d^2=12^2$, so $d=\sqrt{144-w^2}$. Then we may write $S=kwd^2=kw(144-w^2)=144kw-kw^3$ for $0<w<12$, so $\frac{dS}{dw}=144k-3kw^2=-3k(w^2-48)$.
The critical point (for $0 < w < 12$) occurs at $w=\sqrt{48}=4\sqrt{3}$. Since $\frac{dS}{dw}>0$ for $0<w<4\sqrt{3}$ and $\frac{dS}{dw}<0$ for $4\sqrt{3}<w<12$, the critical point corresponds to the maximum strength. The dimensions are $4\sqrt{3}$ in. wide by $4\sqrt{6}$ in. deep.

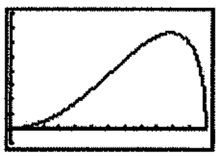
(b)



[0, 12] by [−100, 800]

The graph of $S=144w-w^3$ is shown. The maximum strength shown in the graph occurs at $w=4\sqrt{3}\approx 6.9$, which agrees with the answer to part (a).

(c)



[0, 12] by [−100, 800]

The graph of $S=d^2\sqrt{144-d^2}$ is shown. The maximum strength shown in the graph occurs at $d=4\sqrt{6}\approx 9.8$, which agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.

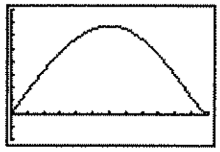
Changing the value of k changes the maximum strength, but not the dimensions of the strongest beam. The graphs for different values of k look the same except that the vertical scale is different.

38. (a) Note that $w^2+d^2=12^2$, so $d=\sqrt{144-w^2}$. Then we may write $S=kwd^3=kw(144-w^2)^{3/2}$, so

$$\frac{dS}{dw}=kw\cdot\frac{3}{2}(144-w^2)^{1/2}(-2w)+k(144-w^2)^{3/2}(1)$$
$$=(k\sqrt{144-w^2})(-3w^2+144-w^2)$$
$$=(-4k\sqrt{144-2^2})(w^2-36)$$

The critical point (for $0 < w < 12$) occurs at $w = 6$. Since $\frac{dS}{dw}>0$ for $0<w<6$ and $\frac{dS}{dw}<0$ for $6<w<12$, the critical point corresponds to the maximum stiffness. The dimensions are 6 in. wide by $6\sqrt{3}$ in. deep.

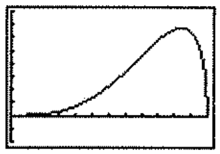
(b)



[0, 12] by [−2000, 8000]

The graph of $S=w(144-w^2)^{3/2}$ is shown. The maximum stiffness shown in the graph occurs at $w = 6$, which agrees with the answer to part (a).

(c)



[0, 12] by [−2000, 8000]

The graph of $S=d^3\sqrt{144-d^2}$ is shown. The maximum stiffness shown in the graph occurs at $d=6\sqrt{3}\approx 10.4$ agrees with the answer to part (a), and its value is the same as the maximum value found in part (b), as expected.
Changing the value of k changes the maximum stiffness, but not the dimensions of the stiffest beam. The graphs for different values of k look the same except that the vertical scale is different.

39. (a) $v(t)=s'(t)=-10\pi\sin\pi t$

The speed at time t is $10\pi|\sin\pi t|$. The maximum speed is 10π cm/sec and it occurs at $t=\frac{1}{2}$, $t=\frac{3}{2}$, $t=\frac{5}{2}$, and $t=\frac{7}{2}$ sec. The position at these times is $s=0$ cm (rest position), and the acceleration $a(t)=v'(t)=-10\pi^2\cos\pi t$ is 0 cm/sec^2 at these times.

39. Continued

(b) Since $a(t) = -10\pi^2 \cos \pi t$, the greatest magnitude of the acceleration occurs at $t = 0$, $t = 1$, $t = 2$, $t = 3$, and $t = 4$. At these times, the position of the cart is either $s = -10$ cm or $s = 10$ cm, and the speed of the cart is 0 cm/sec.

40. Since $\frac{di}{dt} = -2\sin t + 2\cos t$, the largest magnitude of the current occurs when $-2\sin t + 2\cos t = 0$, or $\sin t = \cos t$. Squaring both sides gives $\sin^2 t = \cos^2 t$, and we know that $\sin^2 t + \cos^2 t = 1$, so $\sin^2 t = \cos^2 t = \frac{1}{2}$. Thus the possible values of t are $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$, and so on. Eliminating extraneous solutions, the solutions of $\sin t = \cos t$ are $t = \frac{\pi}{4} + k\pi$ for integers k, and at these times $|i| = |2\cos t + 2\sin t| = 2\sqrt{2}$. The peak current is $2\sqrt{2}$ amps.

41. The square of the distance is

$$D(x) = \left(x - \frac{3}{2}\right)^2 + (\sqrt{x} + 0)^2 = x^2 - 2x + \frac{9}{4},$$

so $D'(x) = 2x - 2$ and the critical point occurs at $x = 1$. Since $D'(x) < 0$ for $x < 1$ and $D'(x) > 0$ for $x > 1$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(1)} = \frac{\sqrt{5}}{2}$.

42. Calculus method:

The square of the distance from the point $(1, \sqrt{3})$ to $(x, \sqrt{16 - x^2})$ is given by

$$\begin{aligned} D(x) &= (x-1)^2 + (\sqrt{16-x^2} - \sqrt{3})^2 \\ &= x^2 - 2x + 1 + 16 - x^2 - 2\sqrt{48 - 3x^2} + 3 \\ &= -2x + 20 - 2\sqrt{48 - 3x^2}. \text{ Then} \end{aligned}$$

$$D'(x) = -2 - \frac{2}{2\sqrt{48-3x^2}}(-6x) = -2 + \frac{6x}{\sqrt{48-3x^2}}.$$

Solving $D'(x) = 0$, we have:

$$\begin{aligned} 6x &= 2\sqrt{48-3x^2} \\ 36x^2 &= 4(48 - 3x^2) \\ 9x^2 &= 48 - 3x^2 \\ 12x^2 &= 48 \\ x &= \pm 2 \end{aligned}$$

We discard $x = -2$ as an extraneous solution, leaving $x = 2$. Since $D'(x) < 0$ for $-4 < x < 2$ and $D'(x) > 0$ for $2 < x < 4$, the critical point corresponds to the minimum distance. The minimum distance is $\sqrt{D(2)} = 2$.

Geometry method:

The semicircle is centered at the origin and has radius 4. The distance from the origin to $(1, \sqrt{3})$ is $\sqrt{1^2 + (\sqrt{3})^2} = 2$. The shortest distance from the point to the semicircle is the distance along the radius containing the point $(1, \sqrt{3})$. That distance is $4 - 2 = 2$.

43. No. Since $f(x)$ is a quadratic function and the coefficient of x^2 is positive, it has an absolute minimum at the point where $f'(x) = 2x - 1 = 0$, and the point is $\left(\frac{1}{2}, \frac{3}{4}\right)$.

44. (a) Because $f(x)$ is periodic with period 2π.

(b) No. Since $f(x)$ is continuous on $[0, 2\pi]$, its absolute minimum occurs at a critical point or endpoint. Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x) = -4\sin x - 2\sin 2x &= 0 \\ -4\sin x - 4\sin x \cos x &= 0 \\ -4(\sin x)(1 + \cos x) &= 0 \\ \sin x &= 0 \text{ or } \cos x = -1 \\ x &= 0, \pi, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 8)$, $(\pi, 0)$, and $(2\pi, 8)$. Thus, $f(x)$ has an absolute minimum at $(\pi, 0)$ and it is never negative.

45. (a)

$$\begin{aligned} 2\sin t &= \sin 2t \\ 2\sin t &= 2\sin t \cos t \\ 2(\sin t)(1 - \cos t) &= 0 \\ \sin t &= 0 \text{ or } \cos t = 1 \end{aligned}$$

$t = k\pi$, where k is an integer

The masses pass each other whenever t is an integer multiple of π seconds.

(b) The vertical distance between the objects is the absolute value of $f(x) = \sin 2t - 2\sin t$.

Find the critical points in $[0, 2\pi]$:

$$\begin{aligned} f'(x) = 2\cos 2t - 2\cos t &= 0 \\ 2(2\cos^2 t - 1) - 2\cos t &= 0 \\ 2(2\cos^2 t - \cos t - 1) &= 0 \\ 2(2\cos t + 1)(\cos t - 1) &= 0 \end{aligned}$$

$$\begin{aligned} \cos t &= -\frac{1}{2} \text{ or } \cos t = 1 \\ t &= \frac{2\pi}{3}, \frac{4\pi}{3}, 0, 2\pi \end{aligned}$$

The critical points (and endpoints) are $(0, 0)$, $\left(\frac{2\pi}{3}, -\frac{3\sqrt{3}}{2}\right)$, $\left(\frac{4\pi}{3}, \frac{3\sqrt{3}}{2}\right)$, and $(2\pi, 0)$

The distance is greatest when $t = \frac{2\pi}{3}$ sec and when $t = \frac{4\pi}{3}$ sec. The distance at those times is $\frac{3\sqrt{3}}{2}$ meters.

46. (a) $\sin t = \sin\left(t+\frac{\pi}{3}\right)$

$$\sin t = \sin t\cos\frac{\pi}{3} + \cos t\sin\frac{\pi}{3}$$
$$\sin t = \frac{1}{2}\sin t + \frac{\sqrt{3}}{2}\cos t$$
$$\frac{1}{2}\sin t = \frac{\sqrt{3}}{2}\cos t$$
$$\tan t = \sqrt{3}$$

Solving for t, the particles meet at $t=\frac{\pi}{3}$ sec and at $t=\frac{4\pi}{3}$ sec.

(b) The distance between the particles is the absolute value of $f(t)=\sin\left(t+\frac{\pi}{3}\right)-\sin t = \frac{\sqrt{3}}{2}\cos t - \frac{1}{2}\sin t$. Find the critical points in $[0, 2\pi]$:

$$f'(t) = -\frac{\sqrt{3}}{2}\sin t - \frac{1}{2}\cos t = 0$$
$$-\frac{\sqrt{3}}{2}\sin t = \frac{1}{2}\cos t$$
$$\tan t = -\frac{1}{\sqrt{3}}$$

The solutions are $t=\frac{5\pi}{6}$ and $t=\frac{11\pi}{6}$, so the critical points are at $\left(\frac{5\pi}{6}, -1\right)$ and $\left(\frac{11\pi}{6}, 1\right)$, and the interval endpoints are at $\left(0, \frac{\sqrt{3}}{2}\right)$, and $\left(2\pi, \frac{\sqrt{3}}{2}\right)$. The particles are farthest apart at $t=\frac{5\pi}{6}$ sec and at $t=\frac{11\pi}{6}$ sec, and the maximum distance between the particles is 1 m.

(c) We need to maximize $f'(t)$, so we solve $f''(t)=0$.

$$f''(t) = -\frac{\sqrt{3}}{2}\cos t + \frac{1}{2}\sin t = 0$$
$$\frac{1}{2}\sin t = \frac{\sqrt{3}}{2}\cos t$$

This is the same equation we solved in part (a), so the solutions are $t=\frac{\pi}{3}$ sec and $t=\frac{4\pi}{3}$ sec.

For the function $y=f'(t)$, the critical points occur at $\left(\frac{\pi}{3}, -1\right)$ and $\left(\frac{4\pi}{3}, 1\right)$, and the interval endpoints are at $\left(0, -\frac{1}{2}\right)$ and $\left(2\pi, -\frac{1}{2}\right)$.

Thus, $|f'(t)|$ is maximized at $t=\frac{\pi}{3}$ and $t=\frac{4\pi}{3}$. But these are the instants when the particles pass each other, so the graph of $y=|f(t)|$ has corners at these points and $\frac{d}{dt}|f(t)|$ is undefined at these instants. We cannot say that the distance is changing the fastest at any particular instant, but we can say that near $t=\frac{\pi}{3}$ or $t=\frac{4\pi}{3}$ the distance is changing faster than at any other time in the interval.

47. The trapezoid has height $(\cos\theta)$ft and the trapezoid bases measure 1 ft and $(1+2\sin\theta)$ft, so the volume is given by

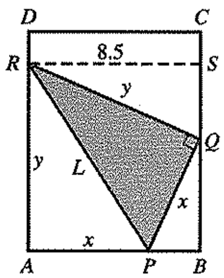
$$V(\theta) = \frac{1}{2}(\cos\theta)(1+1+2\sin\theta)(20)$$
$$= 20(\cos\theta)(1+\sin\theta).$$

Find the critical points for $0 \le \theta < \frac{\pi}{2}$:

$$V'(\theta) = 20(\cos\theta)(\cos\theta) + 20(1+\sin\theta)(-\sin\theta) = 0$$
$$20\cos^2\theta - 20\sin\theta - 20\sin^2\theta = 0$$
$$20(1-\sin^2\theta) - 20\sin\theta - 20\sin^2\theta = 0$$
$$-20(2\sin^2\theta + \sin\theta - 1 = 0$$
$$-20(2\sin\theta - 1)(\sin\theta + 1) = 0$$
$$\sin\theta = \frac{1}{2} \text{ or } \sin\theta = -1$$
$$\theta = \frac{\pi}{6}$$

The critical point is at $\left(\frac{\pi}{6}, 15\sqrt{3}\right)$. Since $V'(\theta) > 0$ for $0 \le \theta < \frac{\pi}{6}$ and $V'(\theta) < 0$ for $\frac{\pi}{6} < \theta < \frac{\pi}{2}$, the critical point corresponds to the maximum possible trough volume. The volume is maximized when $\theta = \frac{\pi}{6}$.

48. (a)



Sketch segment RS as shown, and let y be the length of segment QR. Note that $PB = 8.5 - x$, and so $QB = \sqrt{x^2 - (8.5-x)^2} = \sqrt{8.5(2x-8.5)}$.

Also note that triangles QRS and PQB are similar.

$$\frac{QR}{RS} = \frac{PQ}{QB}$$
$$\frac{y}{8.5} = \frac{x}{\sqrt{8.5(2x-8.5)}}$$

48. Continued

(a) $\frac{y^2}{8.5^2} = \frac{x^2}{8.5(2x-8.5)}$

$$y^2 = \frac{8.5x^2}{2x-8.5}$$
$$L^2 = x^2 + y^2$$
$$L^2 = x^2 + \frac{8.5x^2}{2x-8.5}$$
$$L^2 = \frac{x^2(2x-8.5)+8.5x^2}{2x-8.5}$$
$$L^2 = \frac{2x^3}{2x-8.5}$$

(b) Note that $x > 4.25$, and let $f(x) = L^2 = \frac{2x^2}{2x-8.5}$. Since $y \le 11$, the approximate domain of f is $5.20 \le x \le 8.5$. Then

$$f'(x) = \frac{(2x-8.5)(6x^2)-(2x^3)(2)}{(2x-8.5)^2} = \frac{x^2(8x-51)}{(2x-8.5)^2}$$

For $x > 5.20$, the critical point occurs at $x = \frac{51}{8} = 6.375$ in., and this corresponds to a minimum value of $f(x)$ because $f'(x) < 0$ for $5.20 < x < 6.375$ and $f'(x) > 0$ for $x > 6.375$. Therefore, the value of x that minimizes L^2 is $x = 6.375$ in.

(c) The minimum value of L is

$$\sqrt{\frac{2(6.375)^3}{2(6.375)-8.5}} \approx 11.04 \text{ in.}$$

49. Since $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right) = \frac{C}{2}M^2 - \frac{1}{3}M^3$, we have

$\frac{dR}{dM} = CM - M^2$. Let $f(M) = CM - M^2$. Then $f'(M) = C - 2M$, and the critical point for f occurs at $M = \frac{C}{2}$. This value corresponds to a maximum because $f'(M) > 0$ for $M < \frac{C}{2}$ and $f'(M) < 0$ for $M > \frac{C}{2}$. The value of M that maximizes $\frac{dR}{dM}$ is $M = \frac{C}{2}$.

50. The profit is given by

$$P(x) = (n)(x-c) = a + b(100-x)(x-c)$$
$$= bx^2 + (100+c)bx + (a - 100bc).$$

Then $P'(x) = -2bx + (100+c)b$
$$= b(100+c-2x).$$

The critical point occurs at $x = \frac{100+c}{2} = 50 + \frac{c}{2}$, and this value corresponds to the maximum profit because

$P'(x) > 0$ for $x < 50 + \frac{c}{2}$ and $P'(x) < 0$ for $x > 50 + \frac{c}{2}$.

A selling price of $50 + \frac{c}{2}$ will bring the maximum profit.

51. True. This is guaranteed by the Extreme Value Theorem (Section 4.1).

52. False. For example, consider $f(x) = x^3$ at $c = 0$.

53. D. $f(x) = x^2(60-x)$

$$f'(x) = x^2(-1) + (60-x)(2x)$$
$$= -x^2 + 120x - 2x^2$$
$$= -3x^2 + 120x$$
$$= -3x(x-40)$$

$x = 0$ or $x = 40$

$60 - x = 60$ $\quad 60 - x = 20$

$$x^2(60-x) = 0$$
$$(40)^2(20) = (1600)(20)$$
$$= 32{,}000$$

54. B. Since $f'(x)$ is negative, $f(x)$ is always decreasing, so $f(25) = 3$.

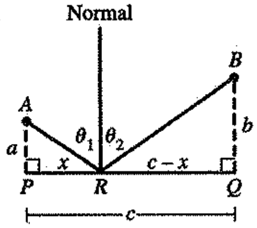
55. B. $A = \frac{1}{2}bh$

$$b^2 + h^2 = 100$$
$$b = \sqrt{100-h^2}$$
$$A = \frac{h}{2}\sqrt{100-h^2}$$
$$A' = \frac{\sqrt{100-h^2}}{2} - \frac{h^2}{2\sqrt{100-h^2}}$$
$$A' = 0 \text{ when } h = \sqrt{50}$$
$$b = \sqrt{100 - \sqrt{50}^2} = \sqrt{50}$$
$$A_{\max} = \frac{1}{2}\sqrt{50}\sqrt{50} = 25$$

56. E. length $= 2x$

$$\text{height} = 30 - x^2 - 4x^2 = 30 - 5x^2$$
$$2x(30-5x^2) = 60x - 10x^3$$
$$\frac{dA}{dx}(60x - 10x^3) = 60 - 30x^2$$
$$x = \sqrt{2}$$
$$2\sqrt{2}(30 - \sqrt{2}^2 - 4(\sqrt{2})^2) = 40\sqrt{2}.$$

57.



Let P be the foot of the perpendicular from A to the mirror, and Q be the foot of the perpendicular from B to the mirror.

57. Continued

Suppose the light strikes the mirror at point R on the way from A to B. Let:

a = distance from A to P
b = distance from B to Q
c = distance from P to Q
x = distance from P to R

To minimize the time is to minimize the total distance the light travels going from A to B. The total distance is

$$D(x)=\sqrt{x^2+a^2}+\sqrt{(c-x)^2+b^2}$$

Then

$$D'(x)=\frac{1}{2\sqrt{x^2+a^2}}(2x)+\frac{1}{2\sqrt{(c-x)^2+b^2}}[-2(c-x)]$$
$$=\frac{x}{\sqrt{x^2+a^2}}-\frac{c-x}{\sqrt{(c-x)^2+b^2}}$$

Solving $D'(x)=0$ gives the equation

$\dfrac{x}{\sqrt{x^2+a^2}}=\dfrac{c-x}{\sqrt{(c-x)^2+b^2}}$ which we will refer to as Equation 1. Squaring both sides, we have:

$$\frac{x^2}{x^2+a^2}=\frac{(c-x)^2}{(c-x)^2+b^2}$$
$$x^2\left[(c-x)^2+b^2\right]=(c-x)^2(x^2+a^2)$$
$$x^2(c-x)^2+x^2b^2=(c-x)^2x^2+(c-x)^2a^2$$
$$x^2b^2=(c-x)^2a^2$$
$$x^2b^2=\left[c^2-2xc+x^2\right]a^2$$
$$0=(a^2-b^2)x^2-2a^2cx+a^2c^2$$
$$0=[(a+b)x-ac][(a-b)x-ac]$$
$$x=\frac{ac}{a+b}\text{ or } x=\frac{ac}{a-b}$$

Note that the value $x=\dfrac{ac}{a-b}$ is an extraneous solution because x and $c-x$ have opposite signs for this value. The only critical point occurs at $x=\dfrac{ac}{a+b}$.

To verify that critical point represents the minimum distance, note that

$$D''(x)=\frac{(\sqrt{x^2+a^2})(1)-(x)\left(\dfrac{x}{\sqrt{x^2+a^2}}\right)}{x^2+a^2}-\frac{(\sqrt{(c-x)^2+b^2})(-1)-(c-x)\left(\dfrac{-(c-x)}{\sqrt{(c-x)^2+b^2}}\right)}{(c-x)^2+b^2}$$
$$=\frac{(x^2+a^2)-x^2}{(x^2+a^2)^{3/2}}-\frac{-[(c-x)^2+b^2]+(c-x)^2}{[(c-x)^2+b^2]^{3/2}}$$
$$=\frac{a^2}{(x^2+a^2)^{3/2}}+\frac{b^2}{[(c-x)^2+b^2]^{3/2}},$$

which is always positive.

We now know that $D(x)$ is minimized when Equation 1 is true, or, equivalently, $\dfrac{PR}{AR}=\dfrac{QR}{BR}$. This means that the two right triangles APR and BQR are similar, which in turn implies that the two angles must be equal.

58. $\dfrac{dv}{dx}=ka-2kx$

The critical point occurs at $x=\dfrac{ka}{2k}=\dfrac{a}{2}$, which represents a maximum value because $\dfrac{d^2v}{dx^2}=-2k$, which is negative for all x. The maximum value of v is

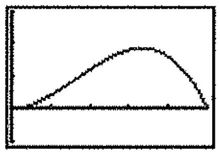
$$kax-kx^2=ka\left(\frac{a}{2}\right)-k\left(\frac{a}{2}\right)^2=\frac{ka^2}{4}.$$

59. (a) $v=cr_0r^2-cr^3$

$$\frac{dv}{dr}=2cr_0r-3cr^2=cr(2r_0-3r)$$

The critical point occurs at $r=\dfrac{2r_0}{3}$. (Note that $r=0$ is not in the domain of v.) The critical point represents a maximum because $\dfrac{d^2v}{dr^2}=2cr_0-6cr=2c(r_0-3r)$, which is negative in the domain $\dfrac{r_0}{2}\le r\le r_0$.

(b) We graph $v=(0.5-r)r^2$, and observe that the maximum indeed occurs at $v=\left(\dfrac{2}{3}\right)0.5=\dfrac{1}{3}$.



[0, 0.5] by [−0.01, 0.03]

60. (a) Since $A''(q)=-kmq^{-2}+\dfrac{h}{2}$, the critical point occurs when $\dfrac{km}{q^2}=\dfrac{h}{2}$, or $q=\dfrac{\sqrt{2km}}{h}$. This corresponds to the minimum value of $A(q)$ because $A''(q)=2kmq^{-3}$, which is positive for $q>0$.

(b) The new formula for average weekly cost is

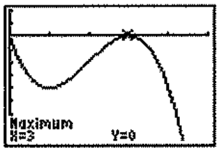
$$B(q)=\frac{(k+bq)m}{q}+cm+\frac{hq}{2}$$
$$=\frac{km}{q}+bm+cm+\frac{hq}{2}$$
$$=A(q)+bm$$

Since $B(q)$ differs from $A(q)$ by a constant, the minimum value of $B(q)$ will occur at the same q-value as the minimum value of $A(q)$. The most economical quantity is again $\dfrac{\sqrt{2km}}{h}$.

61. The profit is given by

$$p(x) = r(x) - c(x)$$
$$= 6x - (x^3 - 6x^2 + 15x)$$
$$= -x^3 + 6x^2 - 9x, \text{ for } x \ge 0.$$

Then $p'(x) = -3x^2 + 12x - 9 = -3(x-1)(x-3)$, so the critical points occur at $x = 1$ and $x = 3$. Since $p'(x) < 0$ for $0 \le x < 1$, $p'(x) > 0$ for $1 < x < 3$, and $p'(x) < 0$ for $x > 3$, the relative maxima occur at the endpoint $x = 0$ and at the critical point $x = 3$. Since $p(0) = p(3) = 0$, this means that for $x \ge 0$, the function $p(x)$ has its absolute maximum value at the points (0, 0) and (3, 0). This result can also be obtained graphically, as shown.



[0, 5] by [−8, 2]

62. The average cost is given by

$a(x) = \dfrac{c(x)}{x} = x^2 - 20x + 20{,}000$. Therefore,

$a'(x) = 2x - 20$ and the critical value is $x = 10$, which represents the minimum because $a''(x) = 2$, which is positive for all x. The average cost is minimized at a production level of 10 items.

63. **(a)** According to the graph, $y'(0) = 0$.

(b) According to the graph, $y'(-L) = 0$.

(c) $y(0) = 0$, so $d = 0$.

Now $y'(x) = 3ax^2 + 2bx + c$, so $y'(0)$ implies that $c = 0$. Therefore, $y(x) = ax^3 + bx^2$ and $y'(x) = 3ax^2 + 2bx$. Then $y(-L) = -aL^3 + bL^2 = H$ and $y'(-L) = 3aL^2 - 2bL = 0$, so we have two linear equations in the two unknowns a and b. The second equation gives $b = \dfrac{3aL}{2}$. Substituting into the first

equation, we have $-aL^3 + \dfrac{3aL^3}{2} = H$, or

$\dfrac{aL^3}{2} = H$, so $a = 2\dfrac{H}{L^3}$. Therefore, $b = 3\dfrac{H}{L^2}$ and the equation for y

is $y(x) = 2\dfrac{H}{L^3}x^3 + 3\dfrac{H}{L^2}x^2$, or

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$

64. **(a)** The base radius of the cone is $r = \dfrac{2\pi a - x}{2\pi}$ and so the height is $h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \left(\dfrac{2\pi a - x}{2\pi}\right)^2}$. Therefore,

$$V(x) = \frac{\pi}{3}r^2 h = \frac{\pi}{3}\left(\frac{2\pi a - x}{2\pi}\right)^2 \sqrt{a^2 - \left(\frac{2\pi a - x}{2\pi}\right)^2}.$$

(b) To simplify the calculations, we shall consider the volume as a function of r:

volume $= f(r) = \dfrac{\pi}{3}r^2\sqrt{a^2 - r^2}$, where $0 < r < a$.

$$f'(r) = \frac{\pi}{3}\frac{d}{dr}\left(r^2\sqrt{a^2 - r^2}\right)$$
$$= \frac{\pi}{3}\left[r^2 \cdot \frac{1}{2\sqrt{a^2 - r^2}} \cdot (-2r) + \left(\sqrt{a^2 - r^2}\right)(2r)\right]$$
$$= \frac{\pi}{3}\left[\frac{-r^3 + 2r(a^2 - r^2)}{\sqrt{a^2 - r^2}}\right]$$
$$= \frac{\pi}{3}\left[\frac{(2a^2 r - 3r^2)}{\sqrt{a^2 - r^2}}\right]$$
$$= \frac{\pi r(2a^2 - 2r^2)}{3\sqrt{a^2 - r^2}}$$

The critical point occurs when $r^2 = \dfrac{2a^2}{3}$, which gives $r = a\sqrt{\dfrac{2}{3}} = \dfrac{a\sqrt{6}}{3}$. Then

$h = \sqrt{a^2 - r^2} = \sqrt{a^2 - \dfrac{2a^2}{3}} = \sqrt{\dfrac{a^2}{3}} = \dfrac{a\sqrt{3}}{3}$. Using

$r = \dfrac{a\sqrt{6}}{3}$ and $h = \dfrac{a\sqrt{3}}{3}$,

we may now find the values of r and h for the given values of a

when $a = 4$: $r = \dfrac{4\sqrt{6}}{3}$, $h = \dfrac{4\sqrt{3}}{3}$;

when $a = 5$: $r = \dfrac{5\sqrt{6}}{3}$, $h = \dfrac{5\sqrt{3}}{3}$;

when $a = 6$: $r = 2\sqrt{6}$, $h = 2\sqrt{3}$;

when $a = 8$: $r = \dfrac{8\sqrt{6}}{3}$, $h = \dfrac{8\sqrt{3}}{3}$

(c) Since $r = \dfrac{a\sqrt{6}}{3}$ and $h = \dfrac{a\sqrt{3}}{3}$, the relationship is $\dfrac{r}{h} = \sqrt{2}$.

65. **(a)** Let x_0 represent the fixed value of x at point P, so that P has coordinates (x_0, a) and let $m = f'(x_0)$ be the slope of line RT. Then the equation of line RT is $y = m(x - x_0) + a$. The y-intercept of this line is $m(0 - x_0) + a = a - mx_0$, and the x-intercept is the solution of $m(x - x_0) + a = 0$, or $x = \dfrac{mx_0 - a}{m}$. Let O designate the origin. Then (Area of triangle RST)

65. Continued

(a) $= 2$ (Area of triangle ORT)

$$= 2 \cdot \frac{1}{2} \text{ (}x\text{-intercept of line } RT\text{) (}y\text{-intercept of line } RT\text{)}$$

$$= 2 \cdot \frac{1}{2}\left(\frac{mx_0 - a}{m}\right)(a - mx_0)$$

$$= -m\left(\frac{mx_0 - a}{m}\right)\left(\frac{mx_0 - a}{m}\right)$$

$$= -\left(\frac{mx_0 - a}{m}\right)^2$$

$$= -m\left(x_0 - \frac{a}{m}\right)^2$$

Substituting x for x_0, $f'(x)$ for m, and $f(x)$ for a, we have $A(x) = -f'(x)\left[x - \dfrac{f(x)}{f'(x)}\right]^2$.

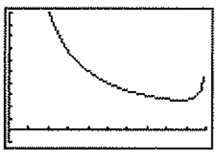
(b) The domain is the open interval (0, 10).

To graph, let $y_1 = f(x) = 5 + 5\sqrt{1 - \dfrac{x^2}{100}}$,

$y_2 = f'(x) = \text{NDER}(y_1)$, and

$$y_3 = A(x) = -y_2\left(x - \frac{y_1}{y^2}\right)^2.$$

The graph of the area function $y_3 = A(x)$ is shown below.



[0, 10] by [–100, 1000]

The vertical asymptotes at $x = 0$ and $x = 10$ correspond to horizontal or vertical tangent lines, which do not form triangles.

(c) Using our expression for the y-intercept of the tangent line, the height of the triangle is

$$a - mx = f(x) - f'(x) \cdot x$$

$$= 5 + \frac{1}{2}\sqrt{100 - x^2} - \frac{-x}{2\sqrt{100 - x^2}}x$$

$$= 5 + \frac{1}{2}\sqrt{100 - x^2} + \frac{x^2}{2\sqrt{100 - x^2}}$$

We may use graphing methods or the analytic method in part (d) to find that the minimum value of $A(x)$ occurs at $x \approx 8.66$. Substituting this value into the expression above, the height of the triangle is 15. This is 3 times the y-coordinate of the center of the ellipse.

(d) Part (a) remains unchanged. The domain is $(0, C)$. To graph, note that

$$f(x) = B + B\sqrt{1 - \frac{x^2}{C^2}} = B + \frac{B}{C}\sqrt{C^2 - x^2} \quad \text{and}$$

$$f'(x) = \frac{B}{C}\frac{1}{2\sqrt{C^2 - x^2}}(-2x) = \frac{-Bx}{C\sqrt{C^2 - x^2}}.$$

Therefore, we have

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{B + \frac{B}{C}\sqrt{C^2 - x^2}}{\frac{-Bx}{C\sqrt{C^2 - x^2}}}\right]^2$$

$$= \frac{Bx}{C\sqrt{C^2 - x^2}}\left[x - \frac{(BC + B\sqrt{C^2 - x^2})\sqrt{C^2 - x^2}}{-Bx}\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + (BC + B\sqrt{C^2 - x^2})(\sqrt{C^2 - x^2})\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[Bx^2 + BC\sqrt{C^2 - x^2} + B(C^2 - x^2)\right]^2$$

$$= \frac{1}{BCx\sqrt{C^2 - x^2}}\left[BC(C + \sqrt{C^2 - x^2})\right]^2$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})^2}{x\sqrt{C^2 - x^2}}$$

$$A'(x) = BC \cdot \frac{(x\sqrt{C^2 - x^2})(2)(C + \sqrt{C^2 - x^2}\left(\frac{-x}{\sqrt{C^2 - x^2}}\right) - (C + \sqrt{C^2 - x^2})^2\left(x\frac{-x}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}(1)\right)}{x^2(C^2 - x^2)}$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)}\left[-2x^2 - (C + \sqrt{C^2 - x^2})\left(\frac{-x^2}{\sqrt{C^2 - x^2}} + \sqrt{C^2 - x^2}\right)\right]$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2\sqrt{C^2 - x^2}}\left[-2x^2 + \frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} + x^2 - (C^2 - x^2)\right]$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)}\left(\frac{Cx^2}{\sqrt{C^2 - x^2}} - C\sqrt{C^2 - x^2} - C^2\right)$$

$$= \frac{BC(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)^{3/2}}\left[Cx^2 - C(C^2 - x^2) - C^2\sqrt{C^2 - x^2}\right]$$

$$= \frac{BC^2(C + \sqrt{C^2 - x^2})}{x^2(C^2 - x^2)^{3/2}}(2x^2 - C^2 - C\sqrt{C^2 - x^2})$$

To find the critical points for $0 < x < C$, we solve:

$$2x^2 - C^2 = C\sqrt{C^2 - x^2}$$

$$4x^4 - 4C^2x^2 + C^4 = C^4 = C^2x^2$$

$$4x^4 - 3C^2x^2 = 0$$

$$x^2(4x^2 - 3C^2) = 0$$

The minimum value of $A(x)$ for $0 < x < C$ occurs at the critical point $x = \dfrac{C\sqrt{3}}{2}$, or $x^2 = \dfrac{3C^2}{4}$. The corresponding triangle height is

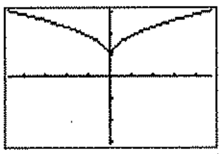
65. Continued

$$a - mx = f(x) - f'(x) \cdot x$$
$$= B + \frac{B}{C}\sqrt{C^2 - x^2} + \frac{Bx^2}{C\sqrt{C^2 - x^2}}$$
$$= B + \frac{B}{C}\sqrt{C^2 - \frac{3C^2}{4}} + \frac{B\left(\frac{3C^2}{4}\right)}{C\sqrt{C^2 - \frac{3C^2}{4}}}$$
$$= B + \frac{B}{C}\left(\frac{C}{2}\right) + \frac{\frac{3BC^2}{4}}{\frac{C^2}{2}}$$
$$= B + \frac{B}{2} + \frac{3B}{2}$$
$$= 3B$$

This shows that the triangle has minimum area when its height is $3B$.

Section 4.5 Linerization and Newton's Method (pp. 233–245)

Exploration 1 Appreciating Local Linearity

1. 

$y = (x^2 + 0.0001)^{1/4} + 0.9$

The function appears to come to a point.

2. $$f^1(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$
$$= \lim_{x \to a} \frac{(x^2 + 0.0001)^{1/4} + 0.9 - ((0 + 0.0001)^{1/4} + 0.9)}{x - 0}$$
$$= \lim_{x \to a} \frac{(x^2 + 0.0001)^{1/4} - 0.1}{x} = 0$$

$f(x)$ is differentiable at $x = 0$, and the equation of the tangent line is $y = 1$.

3. The graph of the function at that point seems to become the graph of a straight line with repeated zooming.

4. The graph will eventually look like the tangent line.

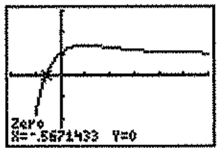
Exploration 2 Using Newton's Method on Your Calculator

See text page 237.

Quick Review 4.5

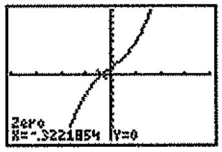
1. $\frac{dy}{dx} = \cos(x^2 + 1) \cdot \frac{d}{dx}(x^2 + 1) = 2x\cos(x^2 + 1)$

2. $$\frac{dy}{dx} = \frac{(x+1)(1 - \sin x) - (x + \cos x)(1)}{(x+1)^2}$$
$$= \frac{x - x\sin x + 1 - \sin x - x - \cos x}{(x+1)^2}$$
$$= \frac{1 - \cos x - (x+1)\sin x}{(x+1)^2}$$

3. 

$[-2, 6]$ by $[-3, 3]$

$x \approx -0.567$

4. 

$[-4, 4]$ by $[-10, 10]$

$x \approx -0.322$

5. $f'(x) = (x)(-e^{-x}) + (e^{-x})(1) = e^{-x} - xe^{-x}$

$f'(0) = 1$

The lines passes through (0, 1) and has slope 1. Its equation is $y = x + 1$.

6. $f'(x) = (x)(-e^{-x}) + (e^{-x})(1) = e^{-x} - xe^{-x}$

$f'(-1) = e^1 - (-e^1) = 2e$

The lines passes through $(-1, -e + 1)$ and has slope $2e$. Its equation is $y = 2e(x + 1) + (-e + 1)$, or $y = 2ex + e + 1$.

7. **(a)** $x + 1 = 0$

$x = -1$

(b) $2ex + e + 1 = 0$

$2ex = -(e + 1)$

$x = -\frac{e+1}{2e} \approx -0.684$

8. $f'(x) = 3x^2 - 4$

$f'(1) = 3(1)^2 - 4 = -1$

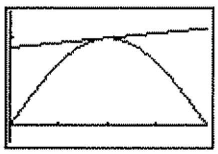
Since $f(1) = -2$ and $f'(1) = -1$, the graph of $g(x)$ passes through $(1, -2)$ and has slope -1. Its equation is $g(x) = -1(x-1) + (-2)$, or $g(x) = -x - 1$.

x	$f(x)$	$g(x)$
0.7	−1.457	−1.7
0.8	−1.688	−1.8
0.9	−1.871	−1.9
1.0	−2	−2
1.1	−2.069	−2.1
1.2	−2.072	−2.2
1.3	−2.003	−2.3

9. $f'(x) = \cos x$

$f'(1.5) = \cos 1.5$

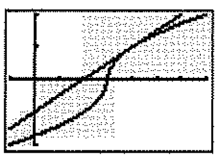
Since $f(1.5) = \sin 1.5$ and $f'(1.5) = \cos 1.5$, the tangent line passes through $(1.5, \sin 1.5)$ and has slope $\cos 1.5$. Its equation is $y = (\cos 1.5)(x - 1.5) + \sin 1.5$, or approximately $y = 0.071x + 0.891$



$[0, \pi]$ by $[-0.2, 1.3]$

10. For $x > 3, f'(x) = \dfrac{1}{2\sqrt{x-3}}$, and so $f'(4) = \dfrac{1}{2}$. Since $f(4) = 1$ and $f'(4) = \dfrac{1}{2}$, the tangent line passes through $(4, 1)$ and has slope $\dfrac{1}{2}$. Its equation is

$y = \dfrac{1}{2}(x-4) + 1$, or $y = \dfrac{1}{2}x - 1$.



$[-1, 7]$ by $[-2, 2]$

Section 4.5 Exercises

1. (a) $f'(x) = 3x^2 - 2$

We have $f(2) = 7$ and $f'(2) = 10$.

$$\begin{aligned} L(x) &= f(2) + f'(2)(x-2) \\ &= 7 + 10(x-2) \\ &= 10x - 13 \end{aligned}$$

(b) Since $f(2.1) = 8.061$ and $L(2.1) = 8$, the approximation differs from the true value in absolute value by less than 10^{-1}.

2. (a) $f'(x) = \dfrac{1}{2\sqrt{x^2+9}}(2x) = \dfrac{x}{\sqrt{x^2+9}}$

We have $f(-4) = 5$ and $f'(-4) = -\dfrac{4}{5}$.

$$\begin{aligned} L(x) &= f(-4) + f'(-4)(x-(-4)) \\ &= 5 - \frac{4}{5}(x+4) \\ &= -\frac{4}{5}x + \frac{9}{5} \end{aligned}$$

(b) Since $f(-3.9) \approx 4.9204$ and $L(-3.9) = 4.92$, the approximation differs from the true value by less than 10^{-3}.

3. (a) $f'(x) = 1 - x^{-2}$

We have $f(1) = 2$ and $f'(1) = 0$.

$$\begin{aligned} L(x) &= f(1) + f'(1)(x-1) \\ &= 2 + 0(x-1) \\ &= 2 \end{aligned}$$

(b) Since $f(1.1)$ 2.009 and $L(1.1) = 2$, the approximation differs from the true value by less than 10^{-2}.

4. (a) $f'(x) = \dfrac{1}{x+1}$

We have $f(0) = 0$ and $f'(0) = 1$.

$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= 0 + 1x \\ &= x \end{aligned}$$

(b) Since $f(0.1) \approx 0.0953$ and $L(0.1) = 0.1$ the approximation differs from the true value by less than 10^{-2}.

5. (a) $f'(x) = \sec^2 x$

We have $f(\pi) = 0$ and $f'(\pi) = 1$.

$$\begin{aligned} L(x) &= f(\pi) + f'(\pi)(x-\pi) \\ &= 0 + 1(x-\pi) \\ &= x - \pi \end{aligned}$$

(b) Since $f(\pi + 0.1) \approx 0.10033$ and $L(\pi + 0.1) = 0.1$, the approximation differs from the true value in absolute value by less than 10^{-3}.

6. (a) $f'(x) = -\dfrac{1}{\sqrt{1-x^2}}$

We have $f(0) = \dfrac{\pi}{2}$ and $f'(0) = -1$.

$$\begin{aligned} L(x) &= f(0) + f'(0)(x-0) \\ &= \frac{\pi}{2} + (-1)(x-0) \\ &= -x + \frac{\pi}{2} \end{aligned}$$

6. Continued

(b) Since $f(0.1) \approx 1.47063$ and $L(0.1) \approx 1.47080$, the approximation differs from the true value in absolute value by less than 10^{-3}.

7. $f'(x) = k(1+x)^{k-1}$

We have $f(0) = 1$ and $f'(0) = k$.

$$L(x) = f(0) + f'(0)(x-0) = 1 + k(x-0) = 1 + kx$$

8. (a) $(1.002)^{100} = (1+0.002)^{100} \approx 1 + (100)(0.002) = 1.2;$

$$\left|1.002^{100} - 1.2\right| \approx 0.021 < 10^{-1}$$

(b) $\sqrt[3]{1.009} = (1+0.009)^{1/3} \approx 1 + \frac{1}{3}(0.009) = 1.003;$

$$\left|\sqrt[3]{1.009} - 1.003\right| \approx 9 \times 10^{-6} < 10^{-5}$$

9. (a) $f(x) = (1-x)^6 = [1+(-x)]^6 \approx 1 + 6(-x) = 1 - 6x$

(b) $f(x) = \dfrac{2}{1-x} = 2[1+(-x)]^{-1} \approx 2[1+(-1)(-x)] = 2 + 2x$

(c) $f(x) = (1+x)^{-1/2} \approx 1 + \left(-\dfrac{1}{2}\right)x = 1 - \dfrac{x}{2}$

10. (a) $f(x) = (4+3x)^{1/3} = 4^{1/3}\left(1 + \dfrac{3x}{4}\right)^{1/3}$

$$\approx 4^{1/3}\left(1 + \frac{1}{3}\left(\frac{3x}{4}\right)\right) = 4^{1/3}\left(1 + \frac{x}{4}\right)$$

(b) $f(x) = \sqrt{2+x^2} = \sqrt{2}\left(1 + \dfrac{x^2}{2}\right)^{1/2}$

$$\approx \sqrt{2}\left(1 + \frac{1}{2}\left(\frac{x^2}{2}\right)\right) = \sqrt{2}\left(1 + \frac{x^2}{4}\right)$$

(c) $f(x) = \left(1 - \dfrac{1}{2+x}\right)^{2/3} = \left[1 + \left(-\dfrac{1}{2+x}\right)\right]^{2/3}$

$$\approx 1 + \frac{2}{3}\left(-\frac{1}{2+x}\right) = 1 - \frac{2}{6+3x}$$

11. $x = 100$

$$f'(100) = \frac{1}{2}(100)^{-1/2} = 0.05$$

$$f(100) = 10 + 0.05(101 - 100) = 10.05$$

12. $x = 27$

$$f'(27) = \frac{1}{3}(27)^{-2/3} = \frac{1}{27}$$

$$f(27) = 3 + (1/27)(26 - 27)$$

$$y = 3 - \frac{1}{27} \approx 2.\overline{962}$$

13. $x = 1000$

$$f'(1000) = \frac{1}{3}(1000)^{-2/3} = \frac{1}{300}$$

$$y = 10 + (1/300)(x - 1000)$$

$$y = 10 - \frac{1}{150} = 9.99\overline{3}$$

14. $x = 81$

$$f'(81) = \frac{1}{2}(81)^{-1/2} = \frac{1}{18}$$

$$y = 9 + \frac{1}{18}(80 - 81)$$

$$y = 9 - \frac{1}{18} = 8.9\overline{4}$$

15. Let $f(x) = x^3 + x - 1$. Then $f'(x) = 3x^2 + 1$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 + x_n - 1}{3x_n^2 + 1}.$$

Note that f is cubic and f' is always positive, so there is exactly one solution. We choose $x_1 = 0$.

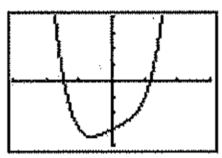
$x_1 = 0$
$x_2 = 1$
$x_3 = 0.75$
$x_4 \approx 0.6860465$
$x_5 \approx 0.6823396$
$x_6 \approx 0.6823278$
$x_7 \approx 0.6823278$

Solution: $x \approx 0.682328$.

16. Let $f(x) = x^4 + x - 3$. Then $f'(x) = 4x^3 + 1$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 + x_n - 3}{4x_n^3 + 1}$$

The graph of $y = f(x)$ shows that $f(x) = 0$ has two solutions.



[−3, 3] by [−4, 4]

$x_1 = -1.5$	$x_1 = 1.2$
$x_2 = -1.455$	$x_2 \approx 1.6541962$
$x_3 \approx -1.4526332$	$x_3 \approx 1.1640373$
$x_4 \approx -1.4526269$	$x_4 \approx 1.1640351$
$x_5 \approx -1.4526269$	$x_5 \approx 1.1640351$

Solution: $x \approx -1.452627,\ 1.164035$

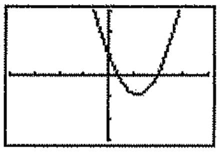
17. Let $f(x) = x^2 - 2x + 1 - \sin x$.

Then $f'(x) = 2x - 2\cos x$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2x_n + 1 - \sin x_n}{2x_n - 2 - \cos x_n}$$

17. Continued

The graph of $y = f(x)$ shows that $f(x) = 0$ has two solutions



[−4, 4] by [−3, 3]

$x_1 = 0.3 \qquad x_1 = 2$
$x_2 \approx 0.3825699 \quad x_2 \approx 1.9624598$
$x_3 \approx 0.3862295 \quad x_3 \approx 1.9615695$
$x_4 \approx 0.3862369 \quad x_4 \approx 1.9615690$
$x_5 \approx 0.3862369 \quad x_5 \approx 1.9615690$

Solutions: $x \approx 0.386237, 1.961569$

18. Let $f(x) = x^4 - 2$. Then $f'(x) = 4x^3$ and

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^4 - 2}{4x_n^3}.$$

Note that $f(x) = 0$ clearly has two solutions, namely $x = \pm\sqrt[4]{2}$. We use Newton's method to find the decimal equivalents.

$x_1 = 1.5$
$x_2 \approx 1.2731481$
$x_3 \approx 1.1971498$
$x_4 \approx 1.1892858$
$x_5 \approx 1.1892071$
$x_6 \approx 1.1892071$
Solutions: $x \approx \pm 1.189207$

19. (a) Since $\frac{dy}{dx} = 3x^2 - 3$, $dy = (3x^2 - 3)\,dx$.

(b) At the given values,
$dy = (3 \cdot 2^2 - 3)(0.05) = 9(0.05) = 0.45.$

20. (a) Since $\frac{dy}{dx} = \frac{(1+x^2)(2) - (2x)(2x)}{(1+x^2)^2} = \frac{2-2x^2}{(1+x^2)^2}$,

$$dy = \frac{2-2x^2}{(1+x^2)^2}dx.$$

(b) At the given values,

$$dy = \frac{2-2(-2)^2}{[1+(-2)^2]^2}(0.1) = \frac{2-8}{5^2}(0.1) = -0.024.$$

21. (a) Since $\frac{dy}{dx} = (x^2)\left(\frac{1}{x}\right) + (\ln x)(2x) = 2x\ln x + x$,
$dy = (2x \ln x + x)dx.$

(b) At the given values,
$dy = [2(1)\ln(1) + 1](0.01) = 1(0.01) = 0.01$

22. (a) Since $\frac{dy}{dx} = (x)\left(\frac{1}{2\sqrt{1-x^2}}\right)(-2x) + (\sqrt{1-x^2})(1)$

$$= \frac{-x^2}{\sqrt{1-x^2}} + \sqrt{1-x^2} = \frac{-x^2 + (1-x^2)}{\sqrt{1-x^2}} = \frac{1-2x^2}{\sqrt{1-x^2}},$$

$$dy = \frac{1-2x^2}{\sqrt{1-x^2}}dx.$$

(b) At the given values, $dy = \frac{1-2(0)^2}{\sqrt{1-(0)^2}}(-0.2) = -0.2.$

23. (a) Since $\frac{dy}{dx} = e^{\sin x}\cos x$, $dy = (\cos x)e^{\sin x}dx.$

(b) At the given values,
$dy = (\cos \pi)(e^{\sin \pi})(-0.1) = (-1)(1)(-0.1) = 0.1.$

24. (a) Since $\frac{dy}{dx} = -3\csc\left(1-\frac{x}{3}\right)\cot\left(1-\frac{x}{3}\right)\left(-\frac{1}{3}\right)$

$$= \csc\left(1-\frac{x}{3}\right)\cot\left(1-\frac{x}{3}\right),$$

$$dy = \csc\left(1-\frac{x}{3}\right)\cot\left(1-\frac{x}{3}\right)dx.$$

(b) At the given values,

$$dy = \csc\left(1-\frac{1}{3}\right)\cot\left(1-\frac{1}{3}\right)(0.1)$$

$$= 0.1\csc\frac{2}{3}\cot\frac{2}{3} \approx 0.205525$$

25. (a) $y + xy - x = 0$
$y(1+x) = x$
$y = \frac{x}{x+1}$

Since $\frac{dy}{dx} = \frac{(x+1)(1) - (x)(1)}{(x+1)^2} = \frac{1}{(x+1)^2}$,

$$dy = \frac{dx}{(x+1)^2}.$$

(b) At the given values,

$$dy = \frac{0.01}{(0+1)^2} = 0.01.$$

26. (a) $2y = x^2 - xy$
$2dy = 2xdx - xdy - ydx$
$dy(2+x) = (2x-y)dx$
$dy = \left(\frac{2x-y}{2+x}\right)dx$

(b) At the given values, and $y = 1$ from the original equation, $dy = \left(\frac{2(2)-1}{2+2}\right)(-0.05) = -0.0375$

27. $\frac{dy}{dx}=\sqrt{1-x^2}$

$dy=\left(-\frac{2x}{2\sqrt{1-x^2}}\right)dx$

$dy=-\frac{x}{\sqrt{1-x^2}}dx$

28. $\frac{dy}{dx}=e^{5x}+x^5$

$dy=(5e^{5x}+5x^4)dx$

29. $\frac{dy}{dx}=\tan^{-1}4x$

$\frac{d}{dx}\tan^{-1}u=\frac{1}{1+u^2}\frac{du}{dx}$

$u=4x$

$\frac{du}{dx}=4$

$dy=\left(\frac{4}{1+16x^2}\right)dx$

30. $\frac{dy}{dx}=(8^x+x^8)$

$\frac{d}{dx}a^x=(\ln a)a^x$

$dy=(8^x\ln 8+8x^7)dx$

31. (a) $\Delta f=f(0.1)-f(0)=0.21-0=0.21$

(b) Since $f'(x)=2x+2, f'(0)=2$.

Therefore, $df=2\,dx=2(0.1)=0.2$.

(c) $|\Delta f-df|=|0.21-0.2|=0.01$

32. (a) $\Delta f=f(1.1)-f(1)=0.231-0=0.231$

(b) Since $f'(x)=3x^2-1, f'(1)=2$.

Therefore, $df=2dx=2(0.1)=0.2$.

(c) $|\Delta f-df|=|0.231-02|=0.031$

33. (a) $\Delta f=f(0.55)-f(0.5)=\frac{20}{11}-2=-\frac{2}{11}$

(b) Since $f'(x)=-x^{-2}, f'(0.5)=-4$.

Therefore, $df=-4\,dx=-4(0.05)=-0.2=-\frac{1}{5}$

(c) $|\Delta f-df|=\left|-\frac{2}{11}+\frac{1}{5}\right|=\frac{1}{55}$

34. (a) $\Delta f=f(1.01)-f(1)=1.04060401-1=0.04060401$

(b) Since $f'(x)=4x^3, f'(1)=4$.

Therefore, $df=4\,dx=4(0.01)=0.04$.

(c) $|\Delta f-df|=|0.04060401-0.04|=0.00060401$

35. Note that $\frac{dV}{dr}=4\pi r^2, dV=4\pi r^2dr$. When r changes from a to $a+dr$, the change in volume is approximately $4\pi a^2dr$.

36. Note that $\frac{dS}{dr}=8\pi r$, so $dS=8\pi r\,dr$. When r changes from a to $a+dr$, the change in surface area is approximately $8\pi a\,dr$.

37. Note that $\frac{dV}{dx}=3x^2$, so $dV=3x^2\,dx$. When x changes from a to $a+dx$, the change in volume is approximately $3a^2\,dx$.

38. Note that $\frac{dS}{dx}=12x$, so $dS=12x\,dx$. When x changes from a to $a+dx$, the change in surface area is approximately $12a\,dx$.

39. Note that $\frac{dV}{dr}=2\pi rh$, so $dV=2\pi rh\,dr$. When r changes from a to $a+dr$, the change in volume is approximately

40. Note that $\frac{dS}{dh}=2\pi r$, so $dS=2\pi r\,dh$. When h changes from a to $a+dh$, the change in lateral surface area is approximately $2\pi r\,dh$.

41. $A=\pi r^2$

$dA=2\pi r\,dr$

$dA=2\pi(10)(0.1)=6.3\text{ in}^2$

42. $V=\frac{4}{3}\pi r^3$

$dV=4\pi r^2dr$

$dV=4\pi(8)^2(0.3)=241\text{ in}^2$

43. $V=s^3$

$dV=3s^2ds$

$dV=3(15)^2(0.2)=135\text{ cm}^2$

44. $A=\frac{\sqrt{3}}{4}s^2$

$dA=\frac{\sqrt{3}}{2}s\,ds$

$dA=\frac{\sqrt{3}}{2}(20)(0.5)=8.7\text{ cm}^2$

45. (a) Note that $f'(0)=\cos 0=1$.

$L(x)=f(0)+f'(0)(x-0)=1+1x=x+1$

(b) $f(0.1)\approx L(0.1)=1.1$

45. Continued

(c) The actual value is less than 1.1. This is because the derivative is decreasing over the interval [0, 0.1], which means that the graph of $f(x)$ is concave down and lies below its linearization in this interval.

46. (a) Note that $A=\pi r^2$ and $\frac{dA}{dr}=2\pi r$, so $dA=2\pi r\,dr$. When r changes from a to $a+dr$, the change in area is approximately $2\pi a\,dr$. Substituting 2 for a and 0.02 for dr, the change in area is approximately $2\pi(2)(0.02)=0.08\pi\approx 0.2513$

(b) $\frac{dA}{A}=\frac{0.08\pi}{4\pi}=0.02=2\%$

47. Let A = cross section area, C = circumference, and D = diameter. Then $D=\frac{C}{\pi}$, so $\frac{dD}{dC}=\frac{1}{\pi}$ and $dD=\frac{1}{\pi}dC$. Also, $A=\pi\left(\frac{D}{2}\right)^2=\pi\left(\frac{C}{2\pi}\right)^2=\frac{C^2}{4\pi}$, so $\frac{dA}{dC}=\frac{C}{2\pi}$ and $dA=\frac{C}{2\pi}dC$. When C increases from 10π in. to $10\pi+2$ in. the diameter increases by $dD=\frac{1}{\pi}(2)=\frac{2}{\pi}\approx 0.6366$ in. and the area increases by approximately $dA=\frac{10\pi}{2\pi}(2)=10\text{ in}^2$.

48. Let x = edge length and V = volume. Then $V=x^3$, and so $dV=3x^2\,dx$. With $x=10$ cm and $dx=0.01x=0.1$ cm, we have $V=10^3=1000\text{ cm}^3$ and $dV=3(10)^2(0.1)=30\text{ cm}^3$, so the percentage error in the volume measurement is approximately $\frac{dV}{V}=\frac{30}{1000}=0.03=3\%$.

49. Let x = side length and A = area. Then $A=x^2$ and $\frac{dA}{dx}=2x$, so $dA=2x\,dx$. We want $|dA|\le 0.02A$, which gives $|2x\,dx|\le 0.02x^2$, or $|dx|\le 0.01x$. The side length should be measured with an error of no more than 1%.

For $\theta=75°=\frac{5\pi}{12}$ radians, we have $|d\theta|<0.04\sin\frac{5\pi}{12}\cos\frac{5\pi}{12}=0.01$ radian. The angle should be measured with an error of less than 0.01 radian (or approximately 0.57 degrees), which is a percentage error of approximately 0.76%.

50. (a) Note that $V=\pi r^2h=10\pi r^2=2.5\pi D^2$, where D is the interior diameter of the tank. Then $\frac{dV}{dD}=5\pi D$, so $dV=5\pi D\,dD$. We want $|dV|\le 0.01V$, which gives $|5\pi D\,dD|\le 0.01(2.5\pi D^2)$, or $|dD|\le 0.005D$. The interior diameter should be measured with an error of no more than 0.5%.

(b) Now we let D represent the *exterior* diameter of the tank, and we assume that the paint coverage rate (number of square feet covered per gallon of paint) is known precisely. Then, to determine the amount of paint within 5%, we need to calculate the lateral surface area S with an error of no more than 5%. Note that $S=2\pi rh=10\pi D$, so $\frac{dS}{dD}=10\pi$ and $dS=10\pi\,dD$. We want $|dS|\le 0.05S$, which gives $|10\pi\,dD|\le 0.05(10\pi D)$, or $dD\le 0.05D$. The exterior diameter should be measured with an error of no more than 5%.

51. Note that $V=\pi r^2h$, where h is constant. Then $\frac{dV}{dr}=2\pi rh$. The percent change is given by $\frac{dV}{V}=\frac{2\pi rh\,dr}{\pi r^2h}=2\frac{dr}{r}=2\frac{0.1\%r}{r}=0.2\%$.

52. Note that $\frac{dV}{dh}=3\pi h^2$, so $dV=3\pi h^2\,dh$. We want $|dV|\le 0.01V$, which gives $|3\pi h^2dh|\le 0.01(\pi h^3)$, or $|dh|\le\frac{0.01h}{3}$. The height should be measured with an error of no more than $\frac{1}{3}\%$.

53. Since $V=\frac{4}{3}\pi r^3$, we have $dV=4\pi r^2dr=4\pi r^2\left(\frac{1}{16\pi}\right)=\frac{r^2}{4}$. The volume error in each case is simply $\frac{r^2}{4}\text{in}^3$.

Sphere Type	True Radius	Tape error	Radius Error	Volume Error
Orange	2 in.	1/8 in.	$1/16\pi$ in.	1 in.^{3}
Melon	4 in.	1/8 in.	$1/16\pi$ in.	4 in.^{3}
Beach Ball	7 in.	1/8 in.	$1/16\pi$ in.	12.25 in.^{3}

54. Since $A = 4\pi r^2$, we have $dA = 8\pi r dr = 8\pi r\left(\frac{1}{16\pi}\right) = \frac{r}{2}$.

The surface area error in each case is simply $\frac{r}{2}$ in^2.

Sphere Type	True Radius	Tape Error	Radius Error	Volume Error
Orange	2 in.	1/8 in.	$1/16\pi$ in.	1in.^{2}
Melon	4 in.	1/8 in.	$1/16\pi$ in.	2in.^{2}
Beach Ball	7 in	1/8 in.	$1/16\pi$ in.	3.5in.^{2}

55. We have $\frac{dW}{dg} = -bg^{-2}$, so $dW = -bg^{-2}dg$.

Then $\frac{dW_{\text{moon}}}{dW_{\text{earth}}} = \frac{-b(5.2)^{-2}dg}{-b(32)^{-2}dg} = \frac{32^2}{5.2^2} \approx 37.87$. The ratio is about 37.87 to 1.

56. **(a)** Note that $T = 2\pi L^{1/2}g^{-1/2}$, so $\frac{dT}{dg} = -\pi L^{1/2}g^{-3/2}$ and

$$dT = -\pi L^{1/2}g^{-3/2}dg.$$

(b) Note that dT and dg have opposite signs. Thus, if g increases, T decreases and the clock speeds up.

(c)
$$-\pi L^{1/2}g^{-3/2}dg = dT$$
$$-\pi(100)^{1/2}(980)^{-3/2}dg = 0.001$$
$$dg \approx -0.9765$$

Since $dg \approx -0.9765, g \approx 980 - 0.9765 = 979.0235$.

57. True. A look at the graph reveals the problem. The graph decreases after $x = 1$ toward a horizontal asymptote of $x = 0$, so the x-intercepts of the tangent lines keep getting bigger without approaching a zero.

58. False. By the product rule, $d(uv) = udv + vdu$.

59. B. $f(x) = e^x$
$$f'(x) = e^x$$
$$L(x) = e^1 + e^1(x-1)$$
$$L(x) = ex$$

60. A. $y = \tan x$
$$dy = (\sec^2 x)dx = (\sec^2 \pi)0.5$$
$$dy = -0.25$$

61. D. $f(x) = x - x^3 + 2$
$$f'(x) = 1 - 3x^2$$
$$x_{n+1} = x_n - \frac{x_n x_n^3 + 2}{1 - 3x_n^2}$$
$$x_2 = 1 - \frac{1 - (1)^3 + 2}{1 - 3(1)^2} = 2$$
$$x_3 = 2 - \frac{2 - (2)^3 + 2}{1 - 3(2)^2} = \frac{18}{11}$$

62. A. $f(x) = \sqrt[3]{x}, x = 64$
$$f'(64) = \frac{1}{3}(64)^{-2/3} = \frac{1}{48}$$
$$\sqrt[3]{66} = 4 + \frac{1}{48}(66 - 64)$$
$$\sqrt[3]{66} = 4.042$$

The calculator returns 4.041, or a 0.01% difference.

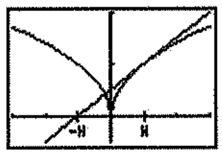
63. If $f'(x) \neq 0$, we have $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)} = x_1 - \frac{0}{f'(x_1)} = x_1$.

Therefore, $x_2 = x_1$, and all later approximations are also equal to x_1.

64. If $x_1 = h$, then $f'(x_1) = \frac{1}{2h^{1/2}}$ and

$x_2 = h - \frac{h^{1/2}}{\frac{1}{2h^{1/2}}} = h - 2h = -h$. If $x_1 = -h$, then

$f'(x_1) = -\frac{1}{2\sqrt{h}}$ and $x_2 = -h - \frac{h^{1/2}}{-\frac{1}{2h^{1/2}}} = -h + 2h = h$

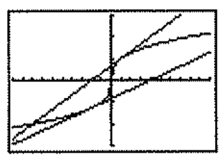


[−3, 3] by [−0.5, 2]

65. Note that $f'(x) = \frac{1}{3}x^{-2/3}$ and so

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^{1/3}}{\frac{x_n^{-2/3}}{3}} = x_n - 3x_n = -2x_n$. For

$x_1 = 1$, we have $x_2 = -2, x_3 = 4, x_4 = -8$, and $x_5 = 16; |x_n| = 2^{n-1}$.



[−10, 10] by [−3, 3]

66. **(a)** **i.** $Q(a) = f(a)$ implies that $b_0 = f(a)$.

ii. Since $Q'(x) = b_1 + 2b_2(x-a), Q'(a) = f'(a)$ implies that $b_1 = f'(a)$.

66. Continued

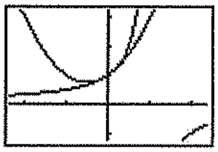
iii. Since $Q''(x) = 2b_2, Q''(a) = f''(a)$ implies that

$$b_2 = \frac{f''(a)}{2}$$

In summary, $b_0 = f(a), b_1 = f'(a)$, and $b_2 = \frac{f''(a)}{2}$.

(b) $f(x) = (1-x)^{-1}$
$f'(x) = -1(1-x)^{-2}(-1) = (1-x)^{-2}$
$f''(x) = -2(1-x)^{-3}(-1) = 2(1-x)^{-3}$
Since $f(0) = 1, f'(0) = 1$, and $f''(0) = 2$, the coefficients are $b_0 = 1, b_1 = 1$, and $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 + x + x^2$.

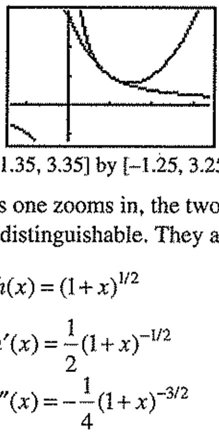
(c)



[–2.35, 2.35] by [–1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(d) $g(x) = x^{-1}$
$g'(x) = -x^{-2}$
$g''(x) = 2x^{-3}$
Since $g(1) = 1, g'(1) = -1$, and $g''(1) = 2$, the coefficients are $b_o = 1, b_1 = -1$, and $b_2 = \frac{2}{2} = 1$. The quadratic approximation is $Q(x) = 1 - (x-1) + (x-1)^2$.



[–1.35, 3.35] by [–1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical

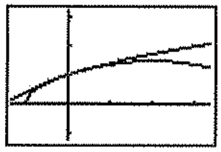
(e) $h(x) = (1+x)^{1/2}$

$h'(x) = \frac{1}{2}(1+x)^{-1/2}$

$h''(x) = -\frac{1}{4}(1+x)^{-3/2}$

Since $h(0) = 1, h'(0) = \frac{1}{2}$, and $h''(0) = -\frac{1}{4}$, the coefficients are $b_0 = 1, b_1 = \frac{1}{2}$, and $b_2 = \frac{-\frac{1}{4}}{2} = -\frac{1}{8}$.

The quadratic approximation is $Q(x) = 1 + \frac{x}{2} - \frac{x^2}{8}$.



[–1.35, 3.35] by [–1.25, 3.25]

As one zooms in, the two graphs quickly become indistinguishable. They appear to be identical.

(f) The linearization of any differentiable function $u(x)$ at $x = a$ is $L(x) = u(a) + u'(a)(x-a) = b_0 + b_1(x-a)$, where b_0 and b_1 are the coefficients of the constant and linear terms of the quadratic approximation. Thus, the linearization for $f(x)$ at $x = 0$ is $1 + x$; the linearization for $g(x)$ at $x = 1$ is $1 - (x-1)$ or $2 - x$; and the linearization for $h(x)$ at $x = 0$ is $1 + \frac{x}{2}$.

67. Finding a zero of sin x by Newton's method would use the recursive formula $x_{n+1} = x_n - \frac{\sin(x_n)}{\cos(x_n)} = x_n - \tan x_n$, and that is exactly what the calculator would be doing. Any zero of sin x would be a multiple of π.

68. Just multiply the corresponding derivative formulas by dx.

(a) Since $\frac{d}{dx}(c) = 0$, $d(c) = 0$.

(b) Since $\frac{d}{dx}(cu) = c\frac{du}{dx}$, $d(cu) = c\,du$.

(c) Since $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$, $d(u+v) = du + dv$

(d) Since $\frac{d}{dx}(u \cdot v) = u\frac{dv}{dx} + v\frac{du}{dx}$, $d(u \cdot v) = u\,dv + v\,du$.

(e) Since $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$, $d\left(\frac{u}{v}\right) = \frac{v\,du - u\,dv}{v^2}$.

(f) Since $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$, $d(u^n) = nu^{n-1}du$.

69. $$\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{\sin x/\cos x}{x}$$
$$= \lim_{x\to 0}\left(\frac{1}{\cos x}\frac{\sin x}{x}\right)$$
$$= \left(\lim_{x\to 0}\frac{1}{\cos x}\right)\left(\lim_{x\to 0}\frac{\sin x}{x}\right)$$
$$= (1)(1) = 1.$$

70. $g(a) = c$, so if $E(a) = 0$, then $g(a) = f(a)$ and $c = f(a)$. Then $E(x) = f(x) - g(x) = f(x) - f(a) - m(x-a)$.

Thus, $\dfrac{E(x)}{x-a} = \dfrac{f(x)-f(a)}{x-a} - m$.

$\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a} = f'(a)$, so $\lim_{x\to a} \dfrac{E(x)}{x-a} = f'(a) - m$.

Therefore, if the limit of $\dfrac{E(x)}{x-a}$ is zero, then $m = f'(a)$ and $g(x) = L(x)$.

71. $f'(x) = \dfrac{1}{2\sqrt{x+1}} + \cos x$

We have $f(0) = 1$ and $f'(0) = \dfrac{3}{2}$

$$L(x) = f(0) + f'(0)(x-0) = 1 + \frac{3}{2}x$$

The linearization is the sum of the two individual linearizations, which are x for $\sin x$ and $1 + \frac{1}{2}x$ for $\sqrt{x+1}$.

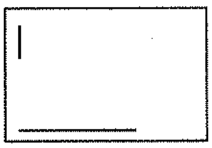
Section 4.6 Related Rates (pp. 246–255)

Exploration 1 Sliding Ladder

1. Here the x-axis represents the ground and the y-axis represents the wall. The curve (x_1, y_1) gives the position of the bottom of the ladder (distance from the wall) at any time t in $0 \le t \le 5$. The curve (x_2, y_2) gives the position of the top of the ladder at any time in $0 \le t \le 5$.

2. $0 \le t \le 5$

4. This is a snapshot at $t \approx 3.1$. The top of the ladder is moving down the y-axis and the bottom of the ladder is moving to the right on the x-axis. The end of the ladder is accelerating. Both axes are hidden from view.



[–1, 15] by [–1, 15]

6. $\dfrac{dy}{dt} = \dfrac{-4T}{\sqrt{10^2 - (2T)^2}}$

7. $y'(3) \approx -4.24$ ft/sec^2. The negative number means the ladder is falling.

8. Since $\lim_{t\to(13/3)^-} y'(t) = -\infty$, the speed of the top of the ladder is infinite as it hits the ground.

Quick Review 4.6

1. $D = \sqrt{(7-0)^2 + (0-5)^2} = \sqrt{49+25} = \sqrt{74}$

2. $D = \sqrt{(b-0)^2 + (0-a)^2} = \sqrt{a^2+b^2}$

3. Use implicit differentiation.

$$\frac{d}{dx}(2xy + y^2) = \frac{d}{dx}(x+y)$$
$$2x\frac{dy}{dx} + 2y(1) + 2y\frac{dy}{dx} = (1) + \frac{dy}{dx}$$
$$(2x + 2y - 1)\frac{dy}{dx} = 1 - 2y$$
$$\frac{dy}{dx} = \frac{1-2y}{2x+2y-1}$$

4. Use implicit differentiation.

$$\frac{d}{dx}(x\sin y) = \frac{d}{dx}(1 - xy)$$
$$(x)(\cos y)\frac{dy}{dx} + (\sin y)(1) = -x\frac{dy}{dx} - y(1)$$
$$(x + x\cos y)\frac{dy}{dx} = -y - \sin y$$
$$\frac{dy}{dx} = \frac{-y-\sin y}{x + x\cos y}$$
$$\frac{dy}{dx} = -\frac{y+\sin y}{x+x\cos y}$$

5. Use implicit differentiation.

$$\frac{d}{dx}x^2 = \frac{d}{dx}\tan y$$
$$2x = \sec^2 y\frac{dy}{dx}$$
$$\frac{dy}{dx} = \frac{2x}{\sec^2 y}$$
$$\frac{dy}{dx} = 2x\cos^2 y$$

6. Use implicit differentiation.

$$\frac{d}{dx}\ln(x+y) = \frac{d}{dx}(2x)$$
$$\frac{1}{x+y}\left(1 + \frac{dy}{dx}\right) = 2$$
$$1 + \frac{dy}{dx} = 2(x+y)$$
$$\frac{dy}{dx} = 2x + 2y - 1$$

7. Using $A(-2, 1)$ we create the parametric equations $x = -2 + at$ and $y = 1 + bt$, which determine a line passing through A at $t = 0$. We determine a and b so that the line passes through $B(4, -3)$ at $t = 1$. Since $4 = -2 + a$, we have $a = 6$, and since $-3 = 1 + b$, we have $b = -4$. Thus, one parametrization for the line segment is $x = -2 + 6t$, $y = 1 - 4t$, $0 \le t \le 1$. (Other answers are possible.)

8. Using $A(0, -4)$, we create the parametric equations $x = 0 + at$ and $y = -4 + bt$, which determine a line passing through A at $t = 0$. We now determine a and b so that the line passes through $B(5, 0)$ at $t = 1$. Since $5 = 0 + a$, we have $a = 5$, and since $0 = -4 + b$, we have $b = 4$. Thus, one parametrization for the line segment is $x = 5t$, $y = -4 + 4t$, $0 \le t \le 1$. (Other answers are possible.)

9. One possible answer: $\frac{\pi}{2} \le t \le \frac{3\pi}{2}$

10. One possible answer: $\frac{3\pi}{2} \le t \le 2\pi$

Section 4.6 Exercises

1. Since $\frac{dA}{dt} = \frac{dA}{dr}\frac{dr}{dt}$, we have $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$,

2. Since $\frac{dS}{dt} = \frac{dS}{dr}\frac{dr}{dt}$, we have $\frac{dS}{dt} = 8\pi r\frac{dr}{dt}$.

3. (a) Since $\frac{dV}{dt} = \frac{dV}{dh}\frac{dh}{dt}$, we have $\frac{dV}{dt} = \pi r^2\frac{dh}{dt}$.

(b) Since $\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt}$, we have $\frac{dV}{dt} = 2\pi rh\frac{dr}{dt}$.

(c)
$$\frac{dV}{dt} = \frac{d}{dt}\pi r^2 h = \pi\frac{d}{dt}(r^2 h)$$
$$\frac{dV}{dt} = \pi\left(r^2\frac{dh}{dt} + h(2r)\frac{dr}{dt}\right)$$
$$\frac{dV}{dt} = \pi r^2\frac{dh}{dt} + 2\pi rh\frac{dr}{dt}$$

4. (a)
$$\frac{dP}{dt} = \frac{d}{dt}(RI^2)$$
$$\frac{dP}{dt} = R\frac{d}{dt}I^2 + I^2\frac{dR}{dt}$$
$$\frac{dP}{dt} = R\left(2I\frac{dI}{dt}\right) + I^2\frac{dR}{dt}$$
$$\frac{dP}{dt} = 2RI\frac{dI}{dt} + I^2\frac{dR}{dt}$$

(b) If P is constant, we have $\frac{dP}{dt} = 0$, which means

$$2RI\frac{dI}{dt} + I^2\frac{dR}{dt} = 0, \text{ or } \frac{dR}{dt} = -\frac{2R}{I}\frac{dI}{dt} = -\frac{2P}{I^3}\frac{dI}{dt}.$$

5.
$$\frac{ds}{dt} = \frac{d}{dt}\sqrt{x^2 + y^2 + z^2}$$
$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\frac{d}{dt}(x^2 + y^2 + z^2)$$
$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt}\right)$$
$$\frac{ds}{dt} = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}}$$

6.
$$\frac{dA}{dt} = \frac{d}{dt}\left(\frac{1}{2}ab\sin\theta\right)$$
$$\frac{dA}{dt} = \frac{1}{2}\left(\frac{da}{dt} \cdot b \cdot \sin\theta + a \cdot \frac{db}{dt} \cdot \sin\theta + ab \cdot \frac{d}{dt}\sin\theta\right)$$
$$\frac{dA}{dt} = \frac{1}{2}\left(b\sin\theta\frac{da}{dt} + a\sin\theta\frac{db}{dt} + ab\cos\theta\frac{d\theta}{dt}\right)$$

7. (a) Since V is increasing at the rate of 1 volt/sec, $\frac{dV}{dt} = 1$ volt/sec.

(b) Since I is decreasing at the rate of $\frac{1}{3}$ amp/sec, $\frac{dI}{dt} = -\frac{1}{3}$ amp/sec.

(c) Differentiating both sides of $V = IR$, we have $\frac{dV}{dt} = I\frac{dR}{dt} + R\frac{dI}{dt}$.

(d) Note that $V = IR$ gives $12 = 2R$, so $R = 6$ ohms. Now substitute the known values into the equation in (c).

$$1 = 2\frac{dR}{dt} + 6\left(-\frac{1}{3}\right)$$
$$3 = 2\frac{dR}{dt}$$
$$\frac{dR}{dt} = \frac{3}{2}\text{ ohms/sec}$$

R is changing at the rate of $\frac{3}{2}$ ohms/sec. Since this value is positive, R is increasing.

8. Step 1:
r = radius of plate
A = area of plate

Step 2:

At the instant in question, $\frac{dr}{dt} = 0.01$ cm/sec, $r = 50$ cm.

Step 3:

We want to find $\frac{dA}{dt}$.

Step 4:

$A = \pi r^2$

Step 5:

$\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$

Step 6:

$\frac{dA}{dt} = 2\pi(50)(0.01) = \pi\text{ cm}^2/\text{sec}$

At the instant in question, the area is increasing at the rate of $\pi\text{cm}^2/\text{sec}$.

9. Step 1:
l = length of rectangle
w = width of rectangle
A = area of rectangle
P = perimeter of rectangle
D = length of a diagonal of the rectangle

Step 2:
At the instant in question,
$\frac{dl}{dt} = -2$ cm/sec, $\frac{dw}{dt} = 2$ cm/sec, $l = 12$ cm, and $w = 5$ cm.
Step 3:
We want to find $\frac{dA}{dt}, \frac{dP}{dt}$, and $\frac{dD}{dt}$.
Steps 4, 5, and 6:

(a) $A = lw$

$$\frac{dA}{dt} = l\frac{dw}{dt} + w\frac{dl}{dt}$$
$$\frac{dA}{dt} = (12)(2) + (5)(-2) = 14 \text{ cm}^2/\text{sec}$$

The rate of change of the area is $14 \text{ cm}^2/\text{sec}$.

(b) $P = 2l + 2w$

$$\frac{dP}{dt} = 2\frac{dl}{dt} + 2\frac{dw}{dt}$$
$$\frac{dP}{dt} = 2(-2) + 2(2) = 0 \text{ cm/sec}$$

The rate of change of the perimeter is 0 cm/sec.

(c) $D = \sqrt{l^2 + w^2}$

$$\frac{dD}{dt} = \frac{1}{2\sqrt{l^2 + w^2}}\left(2l\frac{dl}{dt} + 2w\frac{dw}{dt}\right) = \frac{l\frac{dl}{dt} + w\frac{dw}{dt}}{\sqrt{l^2 + w^2}}$$
$$\frac{dD}{dt} = \frac{(12)(-2) + (5)(2)}{\sqrt{12^2 + 5^2}} = -\frac{14}{13} \text{ cm/sec}$$

The rate of change of the length of the diameter is $-\frac{14}{13}$ cm/sec.

(d) The area is increasing, because its derivative is positive. The perimeter is not changing, because its derivative is zero. The diagonal length is decreasing, because its derivative is negative.

10. Step 1:
x, y, z = edge lengths of the box
V = volume of the box
S = surface area of the box
s = diagonal length of the box

Step 2:
At the instant in question,
$\frac{dx}{dt} = 1$ m/sec, $\frac{dy}{dt} = -2$ m/sec, $\frac{dz}{dt} = 1$ m/sec, $x = 4$m, $y = 3$m, and $z = 2$m.

Step 3:
We want to find $\frac{dV}{dt}, \frac{dS}{dt}$, and $\frac{ds}{dt}$.
Steps 4, 5, and 6:

(a) $V = xyz$

$$\frac{dV}{dt} = xy\frac{dz}{dt} + xz\frac{dy}{dt} + yz\frac{dx}{dt}$$
$$\frac{dV}{dt} = (4)(3)(1) + (4)(2)(-2) + (3)(2)(1) = 2 \text{ m}^3/\text{sec}$$

The rate of change of the volume is $2 \text{ m}^3/\text{sec}$.

(b) $S = 2(xy + xz + yz)$

$$\frac{dS}{dt} 2\left(x\frac{dy}{dt} + y\frac{dx}{dt} + x\frac{dz}{dt} + z\frac{dx}{dt} + y\frac{dz}{dt} + z\frac{dy}{dt}\right)$$
$$\frac{dS}{dt} = 2[(4)(-2) + (3)(1) + (4)(1) + (2)(1) + (2)(1) + (3)(1) + (2)(-2)] = 0 \text{ m}^2/\text{sec}$$

The rate of change of the surface area is $0 \text{ m}^2/\text{sec}$.

(c) $s = \sqrt{x^2 + y^2 + z^2}$

$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2 + y^2 + z^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dy}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt} + z\frac{dz}{dt}}{\sqrt{x^2 + y^2 + z^2}}$$
$$\frac{ds}{dt} = \frac{(4)(1) + (3)(-2) + (2)(1)}{\sqrt{4^2 + 3^2 + 2^2}} = \frac{0}{\sqrt{29}} = 0 \text{ m/sec}$$

The rate of change of the diagonal length is 0 m/sec.

11. Step 1:
r = radius of spherical balloon
S = surface area of spherical balloon
V = volume of spherical balloon

Step 2:
At the instant in question, $\frac{dV}{dt} = 100\pi \text{ ft}^3/\text{min}$ and $r = 5$ ft.

Step 3:
We want to find the values of $\frac{dr}{dt}$ and $\frac{dS}{dt}$.

Steps 4, 5, and 6:

(a) $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$
$$100\pi = 4\pi(5)^2\frac{dr}{dt}$$
$$\frac{dr}{dt} = 1 \text{ ft/min}$$

The radius is increasing at the rate of 1 ft/min.

11. Continued

(b) $S = 4\pi r^2$

$\frac{dS}{dt} = 8\pi r \frac{dr}{dt}$

$\frac{dS}{dt} = 8\pi(5)(1)$

$\frac{dS}{dt} = 40\pi \text{ ft}^2/\text{min}$

The surface area is increasing at the rate of 40π ft^2/min.

12. Step 1:

r = radius of spherical droplet
S = surface area of spherical droplet
V = volume of spherical droplet

Step 2:

No numerical information is given.

Step 3:

We want to show that $\frac{dr}{dt}$ is constant.

Step 4:

$S = 4\pi r^2, V = \frac{4}{3}\pi r^3, \frac{dv}{dt} = kS$ for some constant k

Steps 5 and 6:

Differentiating $V = \frac{4}{3}\pi r^3$, we have $\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$.

Substituting kS for $\frac{dV}{dt}$ and S for $4\pi r^2$, we have $kS = S\frac{dr}{dt}$, or $S\frac{dr}{dt} = k$.

13. Step 1:

s = (diagonal) distance from antenna to airplane
x = horizontal distance from antenna to airplane

Step 2:

At the instant in question,

$s = 10$ mi and $\frac{ds}{dt} = 300$ mph.

Step 3:

We want to find $\frac{dx}{dt}$.

Step 4:

$x^2 + 49 = s^2$ or $x = \sqrt{s^2 - 49}$

Step 5:

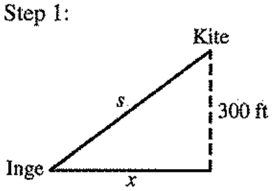
$$\frac{dx}{dt} = \frac{1}{2\sqrt{s^2-49}}\left(2s\frac{ds}{dt}\right) = \frac{s}{\sqrt{s^2-49}}\frac{dt}{dt}$$

Step 6:

$$\frac{dx}{dt} = \frac{10}{\sqrt{10^2-49}}(300) = \frac{3000}{\sqrt{51}} \text{ mph} \approx 420.08 \text{ mph}$$

The speed of the airplane is about 420.08 mph.

14. Step 1:



s = length of kite string
x = horizontal distance from Inge to kite

Step 2:

At the instant in question, $\frac{dx}{dt} = 25$ ft/sec and $s = 500$ ft

Step 3:

We want to find $\frac{ds}{dt}$.

Step 4:

$x^2 + 300^2 = s^2$

Step 5:

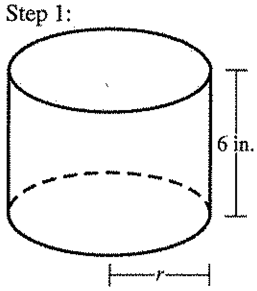
$2x\frac{dx}{dt} = 2s\frac{ds}{dt}$ or $x\frac{dx}{dt} = s\frac{ds}{dt}$

Step 6:

At the instant in question, since $x^2 + 300^2 = s^2$, we have $x = \sqrt{s^2 - 300^2} = \sqrt{500^2 - 300^2} = 400.$

Thus $(400)(25) = (500)\frac{ds}{dt}$, so $\frac{ds}{dt}$, so $\frac{ds}{dt} = 20$ ft/sec. Inge must let the string out at the rate of 20 ft/sec.

15. Step 1:



The cylinder shown represents the shape of the hole.

r = radius of cylinder
V = volume of cylinder

Step 2:

At the instant in question, $\frac{dr}{dt} = \frac{0.001 \text{ in.}}{3 \text{min}} = \frac{1}{3000}$ in./min and (since the diameter is 3.800 in.), $r = 1.900$ in.

Step 3:

We want to find $\frac{dV}{dt}$.

Step 4:

$V = \pi r^2(6) = 6\pi r^2$

Step 5:

$\frac{dV}{dt} = 12\pi r \frac{dr}{dt}$

15. Continued

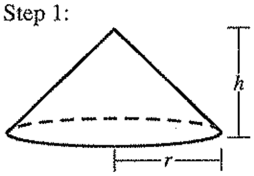
Step 6:

$\frac{dV}{dt} = 12\pi(1.900)\left(\frac{1}{3000}\right) = \frac{19\pi}{2500} = 0.0076\pi$

$\approx 0.0239 \text{ in}^3/\text{min}.$

The volume is increasing at the rate of approximately $0.0239 \text{ im}^3/\text{min}$.

16. Step 1:



r = base radius of cone
h = height of cone
V = volume of cone

Step 2:

At the instant in question, $h = 4$ m and $\frac{dV}{dt} = 10 \text{ m}^3/\text{min}$.

Step 3:

We want to find $\frac{dh}{dt}$ and $\frac{dr}{dt}$.

Step 4:

Since the height is $\frac{3}{8}$ of the base diameter, we have

$h = \frac{3}{8}(2r)$ or $r = \frac{4}{3}h.$

We also have $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{4}{3}h\right)^2 h = \frac{16\pi h^3}{27}$. We will use the equations $V = \frac{16\pi h^3}{27}$ and $r = \frac{4}{3}h$.

Step 5 and 6:

(a) $\frac{dV}{dt} = \frac{16\pi h^2}{9}\frac{dh}{dt}$

$10 = \frac{16\pi(4)^2}{9}\frac{dh}{dt}$

$\frac{dh}{dt} = \frac{45}{128\pi} \text{ m/min} = \frac{1125}{32\pi} \text{ cm/min}$

The height is changing at the rate of $\frac{1125}{32\pi} \approx 11.19 \text{ cm/min}$.

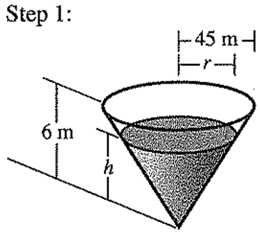
(b) Using the results from Step 4 and part (a), we have

$\frac{dr}{dt} = \frac{4}{3}\frac{dh}{dt} = \frac{4}{3}\left(\frac{1125}{32\pi}\right) = \frac{375}{8\pi} \text{ cm/min}.$

The radius is changing at the rate of

$\frac{375}{8\pi} \approx 14.92 \text{ cm/min}.$

17. Step 1:



r = radius of top surface of water
h = depth of water in reservoir
V = volume of water in reservoir

Step 2:

At the instant in question, $\frac{dV}{dt} = -50 \text{ m}^3/\text{min}$ and $h = 5$ m.

Step 3:

We want to find $-\frac{dh}{dt}$ and $\frac{dr}{dt}$.

Step 4:

Note that $\frac{h}{r} = \frac{6}{45}$ by similar cones, so $r = 7.5h$.

Then $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(7.5h)^2 h = 18.75\pi h^3$

Steps 5 and 6:

(a) Since $V = 18.75\pi h^3, \frac{dV}{dt} = 56.25\pi h^2\frac{dh}{dt}$.

Thus $-50 = 56.25\pi(5^2)\frac{dh}{dt}$, and

so $\frac{dh}{dt} = -\frac{8}{225\pi} \text{ m/min} = -\frac{32}{9\pi} \text{ cm/min}.$

The water level is falling by $\frac{32}{9\pi} \approx 1.13 \text{ cm/min}$.

(Since $\frac{dh}{dt} < 0$, the rate at which the water level is *falling* is positive.)

(b) Since $r = 7.5h, \frac{dr}{dt} = 7.5\frac{dh}{dt} = -\frac{80}{3\pi}$ cm/min. The rate of change of the radius of the water's surface is

$-\frac{80}{3\pi} \approx -8.49 \text{ cm/min}.$

18. (a) Step 1:

y = depth of water in bowl
V = volume of water in bowl

Step 2:

At the instant in question, $\frac{dV}{dt} = -6 \text{ m}^3/\text{min}$ and $y = 8$ m.

Step 3:

We want to find the value of $\frac{dy}{dt}$.

Step 4:

$V = \frac{\pi}{3}y^2(39 - y)$ or $V = 13\pi y^2 - \frac{\pi}{3}y^3$

18. Continued

Step 5:

$$\frac{dV}{dt}=(26\pi y-\pi y^2)\frac{dy}{dt}$$

Step 6:

$$-6=\left[26\pi(8)-\pi(8^2)\right]\frac{dy}{dt}$$

$$-6=144\pi\frac{dy}{dt}$$

$$\frac{dy}{dt}=-\frac{1}{24\pi}\approx-0.01326 \text{ m/min}$$

$$\text{or } -\frac{25}{6\pi}\approx-1.326 \text{ cm/min}$$

(b) Since $r^2+(13-y)^2=13^2$,

$$r=\sqrt{169-(13-y)^2}=\sqrt{26y-y^2}.$$

(c) Step 1:

y = depth of water
r = radius of water surface
V = volume of water in bowl

Step 2:

At the instant in question, $\frac{dV}{dt}=-6 \text{ m}^3/\text{min},\ y=8$ m,

and therefore (from part (a)) $\frac{dy}{dt}=-\frac{1}{24\pi}$ m/min.

Step 3:

We want to find the value of $\frac{dr}{dt}$.

Step 4:

From part (b), $r=\sqrt{26y-y^2}$.

Step 5:

$$\frac{dr}{dt}=\frac{1}{2\sqrt{26y-y^2}}(26-2y)\frac{dy}{dt}=\frac{13-y}{\sqrt{26y-y^2}}\frac{dy}{dt}$$

Step 6:

$$\frac{dr}{dt}=\frac{13-8}{\sqrt{26(8)-8^2}}\left(\frac{1}{-24\pi}\right)=\frac{5}{12}\left(-\frac{1}{24\pi}\right)$$

$$=-\frac{5}{288\pi}\approx-0.00553 \text{ m/min}$$

$$\text{or } -\frac{125}{72\pi}\approx-0.553 \text{ cm/min}$$

19. Step 1:

x = distance from wall to base of ladder
y = height of top of ladder
A = area of triangle formed by the ladder, wall, and ground
θ = angle between the ladder and the ground

Step 2:

At the instant in question, $x=12$ ft and $\frac{dx}{dt}=5$ ft/sec.

Step 3:

We want to find $-\frac{dy}{dy},\frac{dA}{dt}$, and $\frac{d\theta}{dt}$.

Steps 4, 5, and 6:

(a) $x^2+y^2=169$

$$2x\frac{dx}{dt}+2y\frac{dy}{dt}=0$$

To evaluate, note that, at the instant in question,

$$y=\sqrt{169-x^2}=\sqrt{169-12^2}=5.$$

Then $2(12)(5)+2(5)\frac{dy}{dt}=0$

$$\frac{dy}{dt}=-12 \text{ ft/sec}\left(\text{or } -\frac{dy}{dt}=12 \text{ ft/sec}\right)$$

The top of the ladder is sliding down the wall at the rate of 12 ft/sec. (Note that the *downward* rate of motion is positive.)

(b) $A=\frac{1}{2}xy$

$$\frac{dA}{dt}=\frac{1}{2}\left(x\frac{dy}{dt}+y\frac{dx}{dt}\right)$$

Using the results from step 2 and from part (a), we have $\frac{dA}{dt}=\frac{1}{2}[(12)(-12)+(5)(5)]=-\frac{119}{2}$ ft^2/sec. The area of the triangle is changing at the rate of -59.5 ft^2/sec.

(c) $\tan\theta=\frac{y}{x}$

$$\sec^2\theta\frac{d\theta}{dt}=\frac{x\frac{dy}{dt}-y\frac{dx}{dt}}{x^2}$$

Since $\tan\theta=\frac{5}{12}$, we have

$$\left(\text{for } 0\le\theta<\frac{\pi}{2}\right)\cos\theta=\frac{12}{13} \text{ and so } \sec^2\theta\frac{1}{\left(\frac{12}{13}\right)^2}=\frac{169}{144}.$$

Combining this result with the results from step 2 and from part (a), we have $\frac{169}{144}\frac{d\theta}{dt}=\frac{(12)(-12)-(5)(5)}{12^2}$, so $\frac{d\theta}{dt}=-1$ radian/sec. The angle is changing at the rate of -1 radian/sec.

20. Step 1:

h = height (or depth) of the water in the trough
V = volume of water in the trough

Step 2:

At the instant in question, $\frac{dV}{dt}=2.5 \text{ ft}^3/\text{min}$ and $h=2$ ft.

20. Continued

Step 3:

We want to find $\frac{dh}{dt}$.

Step 4:

The width of the top surface of the water is $\frac{4}{3}h$, so we have $V = \frac{1}{2}(h)\left(\frac{4}{3}h\right)(15)$, or $V = 10h^2$

Step 5:

$$\frac{dV}{dt} = 20h\frac{dh}{dt}$$

Step 6:

$$2.5 = 20(2)\frac{dh}{dt}$$

$$\frac{dh}{dt} = 0.0625 = \frac{1}{16} \text{ ft/min}$$

The water level is increasing at the rate of $\frac{1}{16}$ ft/min.

21. Step 1:

l = length of rope
x = horizontal distance from boat to dock
θ = angle between the rope and a vertical line

Step 2:

At the instant in question, $\frac{dl}{dt} = -2$ ft/sec and $l = 10$ ft.

Step 3:

We want to find the values of $-\frac{dx}{dt}$ and $\frac{d\theta}{dt}$.

Steps 4, 5, and 6:

(a) $x = \sqrt{l^2 - 36}$

$$\frac{dx}{dt} = \frac{l}{\sqrt{l^2-36}}\frac{dl}{dt}$$

$$\frac{dx}{dt} = \frac{10}{\sqrt{10^2-36}}(-2) = -2.5 \text{ ft/sec}$$

The boat is approaching the dock at the rate of 2.5 ft/sec.

(b) $\theta = \cos^{-1}\frac{6}{l}$

$$\frac{d\theta}{dt} = -\frac{1}{\sqrt{1-\left(\frac{6}{l}\right)^2}}\left(-\frac{6}{l^2}\right)\frac{dl}{dt}$$

$$\frac{d\theta}{dt} = -\frac{1}{\sqrt{1-0.6^2}}\left(-\frac{6}{10^2}\right)(-2) = -\frac{3}{20} \text{ radian/sec}$$

The rate of change of angle θ is $-\frac{3}{20}$ radian/sec.

22. Step 1:

x = distance from origin to bicycle
y = height of balloon (distance from origin to balloon)
s = distance from balloon to bicycle

Step 2:

We know that $\frac{dy}{dt}$ is a constant 1 ft/sec and $\frac{dx}{dt}$ is a constant 17 ft/sec. Three seconds before the instant in question, the values of x and y are $x = 0$ ft and $y = 65$ ft. Therefore, at the instant in question $x = 51$ ft and $y = 68$ ft.

Step 3:

We want to find the value of $\frac{ds}{dt}$ at the instant in question.

Step 4:

$$s = \sqrt{x^2 + y^2}$$

Step 5:

$$\frac{ds}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$$

Step 6:

$$\frac{ds}{dt} = \frac{(51)(17) + (68)(1)}{\sqrt{51^2 + 68^2}} = 11 \text{ ft/sec}$$

The distance between the balloon and the bicycle is increasing at the rate of 11 ft/sec.

23. $\frac{dy}{dt} = \frac{dy}{dt}\frac{dx}{dt} = -10(1+x^2)^{-2}(2x)\frac{dx}{dt} = -\frac{20x}{(1+x^2)}\frac{dx}{dt}$

Since $\frac{dx}{dt} = 3$cm/sec, we have

$$\frac{dy}{dt} = -\frac{60x}{(1+x^2)^2} \text{ cm/sec.}$$

(a) $\frac{dy}{dt} = -\frac{60(-2)}{[1+(-2)^2]^2} = \frac{120}{5^2} = \frac{24}{5}$ cm/sec

(b) $\frac{dy}{dt} = -\frac{60(0)}{(1+0^2)^2} = 0$ cm/sec

(c) $\frac{dy}{dt} = -\frac{60(20)}{(1+20^2)^2} \approx -0.00746$ cm/sec

24. $\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = (3x^2 - 4)\frac{dx}{dt}$

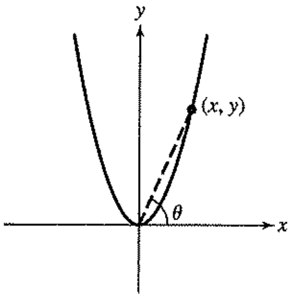
Since $\frac{dx}{dt} = -2$ cm/sec, we have $\frac{dy}{dt} = 8 - 6x^2$ cm/sec.

(a) $\frac{dy}{dt} = 8 - 6(-3)^2 = -46$ cm/sec

(b) $\frac{dy}{dt} = 8 - 6(1)^2 = 2$ cm/sec

(c) $\frac{dy}{dt} = 8 - 6(4)^2 = -88$ cm/sec

25. Step 1:



x = x-coordinate of particle's location
y = y-coordinate of particle's location
θ = angle of inclination of line joining the particle to the origin.

Step 2:
At the instant in question,
$\frac{dx}{dt} = 10$ m/sec and $x = 3$ m.

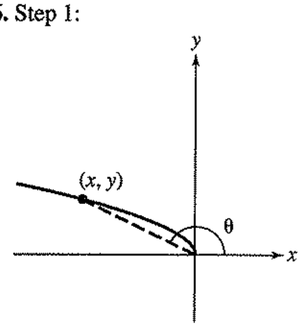
Step 3:
We want to find $\frac{d\theta}{dt}$.

Step 4:
Since $y = x^2$, we have $\tan\theta = \frac{y}{x} = \frac{x^2}{x} = x$ and so, for $x > 0$,
$\theta = \tan^{-1} x$.

Step 5:
$\frac{d\theta}{dt} = \frac{1}{1+x^2}\frac{dx}{dt}$

Step 6:
$\frac{d\theta}{dt} = \frac{1}{1+3^2}(10) = 1$ radian/sec
The angle of inclination is increasing at the rate of 1 radian/sec.

26. Step 1:



x = x-coordinate of particle's location
y = y-coordinate of particle's location
θ = angle of inclination of line joining the particle to the origin

Step 2:
At the instant in question, $\frac{dx}{dt} = -8$ m / sec and $x = -4$ m.

Step 3:
We want to find $\frac{d\theta}{dt}$,

Step 4:
Since $y = \sqrt{-x}$, we have $\tan\theta = \frac{y}{x} = \frac{\sqrt{-x}}{x} = (-x)^{-1/2}$,
and so, for $x < 0$,
$\theta = \pi + \tan^{-1}[-(-x)^{1/2}] = \pi - \tan^{-1}(-x)^{-1/2}$.

Step 5:

$$\frac{d\theta}{dt} = -\frac{1}{1+[(-x)^{-1/2}]^2}\left((-\frac{1}{2}(-x)^{-3/2}(-1)\right)\frac{dx}{dt}$$
$$= -\frac{1}{1-\left(\frac{1}{x}\right)}\frac{1}{2(-x)^{3/2}}\frac{dx}{dt}$$
$$= \frac{1}{2\sqrt{-x}(x-1)}\frac{dx}{dt}$$

Step 6:
$\frac{d\theta}{dt} = \frac{1}{2\sqrt{4}(-4-1)}(-8) = \frac{2}{5}$ radian/sec
The angle of inclination is increasing at the rate of $\frac{2}{5}$ radian/sec.

27. Step 1:
r = radius of balls plus ice
S = surface area of ball plus ice
V = volume of ball plus ice

Step 2:
At the instant in question,
$\frac{dV}{dt} = -8$ mL/min $= -8\text{ cm}^3$/min and $r = \frac{1}{2}(20) = 10$ cm.

Step 3:
We want to find $-\frac{dS}{dt}$.

Step 4:
We have $V = \frac{4}{3}\pi r^3$ and $S = 4\pi r^2$. These equations can be combined by noting that $r = \left(\frac{3V}{4\pi}\right)^{1/3}$, so $S = 4\pi\left(\frac{3V}{4\pi}\right)^{2/3}$

Step 5:
$$\frac{dS}{dt} = 4\pi\left(\frac{2}{3}\right)\left(\frac{3V}{4\pi}\right)^{-1/3}\left(\frac{3}{4\pi}\right)\frac{dV}{dt} = 2\left(\frac{3V}{4\pi}\right)^{-1/3}\frac{dV}{dt}$$

Step 6:
Note that $V = \frac{4}{3}\pi(10)^3 = \frac{4000\pi}{3}$.
$$\frac{dS}{dt} = 2\left(\frac{3}{4\pi}\cdot\frac{4000\pi}{3}\right)^{-1/3}(-8) = \frac{-16}{\sqrt[3]{1000}} = -1.6\text{ cm}^2/\text{min}$$
Since $\frac{dS}{dt} < 0$, the rate of *decrease* is positive. The surface area is decreasing at the rate of 1.6 cm^2/min.

28. Step 1:
x = x-coordinate of particle
y = y-coordinate of particle
D = distance from origin to particle

Step 2:
At the instant in question, $x = 5$ m, $y = 12$ m,
$\frac{dx}{dt} = -1$ m/sec, and $\frac{dy}{dt} = -5$ m/sec.

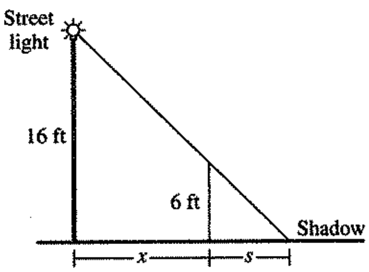
Step 3:
We want to find $\frac{dD}{dt}$.

Step 4:
$D = \sqrt{x^2 + y^2}$

Step 5:
$$\frac{dD}{dt} = \frac{1}{2\sqrt{x^2+y^2}}\left(2x\frac{dx}{dt} + 2y\frac{dy}{dt}\right) = \frac{x\frac{dx}{dt} + y\frac{dy}{dt}}{\sqrt{x^2+y^2}}$$

Step 6:
$$\frac{dD}{dt} = \frac{(5)(-1) + (12)(-5)}{\sqrt{5^2 + 12^2}} = -5 \text{ m/sec}$$
The particle's distance from the origin is changing at the rate of -5 m/sec.

29. Step 1:



x = distance from streetlight base to man
s = length of shadow

Step 2:
At the instant in question, $\frac{dx}{dt} = -5$ ft/sec and $x = 10$ ft.

Step 3:
We want to find $\frac{ds}{dt}$.

Step 4:
By similar triangles, $\frac{s}{6} = \frac{s+x}{16}$. This is equivalent to
$16s = 6s + 6x$, or $s = \frac{3}{5}x$.

Step 5:
$$\frac{ds}{dt} = \frac{3}{5}\frac{dx}{dt}$$

Step 6:
$$\frac{ds}{dt} = \frac{3}{5}(-5) = -3 \text{ ft/sec}$$
The shadow length is changing at the rate of -3 ft/sec.

30. Step 1:
s = distance ball has fallen
x = distance from bottom of pole to shadow

Step 2:
At the instant in question, $s = 16\left(\frac{1}{2}\right)^2 = 4$ ft and
$\frac{ds}{dt} = 32\left(\frac{1}{2}\right) = 16$ ft/sec.

Step 3:
We want to find $\frac{dx}{dt}$.

Step 4:
By similar triangles, $\frac{x-30}{50-s} = \frac{x}{50}$. This is equivalent to
$50x - 1500 = 50x - sx$, or $sx = 1500$. We will use
$x = 1500s^{-1}$.

Step 5 :
$$\frac{dx}{dt} = -500s^{-2}\frac{ds}{dt}$$

Step 6:
$$\frac{dx}{dt} = -1500(4)^{-2}(16) = -1500 \text{ ft/sec}$$
The shadow is moving at a velocity of -1500 ft/sec.

31. Step 1:
x = position of car ($x = 0$ when car is right in front of you)
θ = camera angle. (We assume θ is negative until the car passess in front of you, and then positive.)

Step 2:
At the first instant in question, $x = 0$ ft and $\frac{dx}{dt} = 264$ ft/sec.
A half second later, $x\frac{1}{2}(264) = 132$ ft and $\frac{dx}{dt} = 264$ ft/sec.

Step 3:
We want to find $\frac{d\theta}{dt}$ at each of the two instants.

Step 4:
$$\theta = \tan^{-1}\left(\frac{x}{132}\right)$$

Step 5:
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{x}{132}\right)^2} \cdot \frac{1}{132}\frac{dx}{dt}$$

31. Continued

Step 6:

When $x = 0: \frac{d\theta}{dt} = \frac{1}{1+\left(\frac{0}{132}\right)^2}\left(\frac{1}{132}\right)(264) = 2$ radians/sec

When $x = 132: \frac{d\theta}{dt} = \frac{1}{1+\left(\frac{132}{132}\right)^2}\left(\frac{1}{132}\right)(264) = 1$ radians/sec

32. Step 1:
p = x-coordinate of plane's position
x = x-coordinate of car's position
s = distance from plane to car (line-of-sight)

Step 2:
At the instant in question,

$p = 0, \frac{dp}{dt} = 120\,\text{mph},\ s = 5\,\text{mi, and}\ \frac{ds}{dt} = -160\,\text{mph}.$

Step 3:

We want to find $-\frac{dx}{dt}$.

Step 4:

$(x-p)^2 + 3^2 = s^2$

Step 5:

$2(x-p)\left(\frac{dx}{dt} - \frac{dp}{dt}\right) = 2s\frac{ds}{dt}$

Step 6:
Note that, at the instant in question,
$x = \sqrt{5^2 - 3^2} = 4\,\text{mi}.$

$$2(4-0)\left(\frac{dx}{dt} - 120\right) = 2(5)(-160)$$
$$8\left(\frac{dx}{dt} - 120\right) = -1600$$
$$\frac{dx}{dt} - 120 = -200$$
$$\frac{dx}{dt} = -80\,\text{mph}$$

The car's speed is 80 mph.

33. Step 1:
s = shadow length
θ = sun's angle of elevation

Step 2:
At the instant in question,

$s = 60$ ft and $\frac{d\theta}{dt} = 0.27°/\text{min} = 0.0015\pi$ radian/min.

Step 3:

We want to find $-\frac{ds}{dt}$.

Step 4:

$\tan\theta = \frac{80}{s}$ or $s = 80\cot\theta$

Step 5:

$\frac{ds}{dt} = -80\csc^2\theta\frac{d\theta}{dt}$

Step 6:
Note that, at the moment in question, since tan $\theta = \frac{80}{60}$ and $0 < \theta < \frac{\pi}{2}$, we have $\sin\theta = \frac{4}{5}$ and so $\csc\theta = \frac{5}{4}$.

$$\frac{ds}{dt} = -80\left(\frac{5}{4}\right)^2(0.0015\pi)$$
$$= -0.1875\pi\frac{\text{ft}}{\text{min}}\cdot\frac{12\,\text{in}}{1\,\text{ft}}$$
$$= -2.25\pi\ \text{in./min}$$
$$\approx -7.1\ \text{in./min}$$

Since $\frac{ds}{dt} < 0$, the rate at which the shadow length is *decreasing* is positive. The shadow length is decreasing at the rate of approximately 7.1 in./min.

34. Step 1:
a = distance from origin to A
b = distance from origin to B
θ = angle shown in problem statement

Step 2:

At the instant in question, $\frac{da}{dt} = -2\,\text{m/sec}, \frac{db}{dt} = 1$ m/sec, $a = 10$m, and $b = 20$m.

Step 3:

We want to find $\frac{d\theta}{dt}$.

Step 4:

$\tan\theta = \frac{a}{b}$ or $\theta = \tan^{-1}\left(\frac{a}{b}\right)$

Step 5:

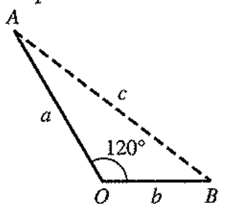
$$\frac{d\theta}{dt} = \frac{1}{1+\left(\frac{a}{b}\right)^2}\,\frac{b\frac{da}{dt} - a\frac{db}{dt}}{b^2} = \frac{b\frac{da}{dt} - a\frac{db}{dt}}{a^2+b^2}$$

Step 6:

$$\frac{d\theta}{dt} = \frac{(20)(-2)-(10)(1)}{10^2+20^2} = -0.1\,\text{radian/sec}$$
$$\approx -5.73\ \text{degrees/sec}$$

To the nearest degree, the angle is changing at the rate of −6 degrees per second.

35. Step 1:



a = distance from O to A
b = distance from O to B
c = distance from A to B

Step 2:
At the instant in question, $a = 5$ nautical miles, $b = 3$ nautical miles, $\frac{da}{dt} = 14$ knots, and $\frac{db}{dt} = 21$ knots.

Step 3:
We want to find $\frac{dc}{dt}$,

Step 4:
Law of Cosines : $c^2 = a^2 + b^2 - 2ab\cos 120°$

$$c^2 = a^2 + b^2 + ab$$

Step 5:

$$2c\frac{dc}{dt} = 2a\frac{da}{dt} + 2b\frac{db}{dt} + a\frac{db}{dt} + b\frac{da}{dt}$$

Step 6:
Note that, at the instant in question,

$$c = \sqrt{a^2 + b^2 + ab} = \sqrt{(5)^2 + (3)^2 + (5)(3)} = \sqrt{49} = 7$$

$$2(7)\frac{dc}{dt} = 2(5)(14) + 2(3)(21) + (5)(21) + (3)(14)$$

$$14\frac{dc}{dt} = 413$$

$$\frac{dc}{dt} = 29.5 \text{ knots}$$

The ships are moving apart at a rate of 29.5 knots.

36. True. Since $\frac{dC}{dt} = 2\pi\frac{dr}{dt}$, a constant $\frac{dr}{dt}$ results in a constant $\frac{dC}{dt}$.

37. False. Since $\frac{dA}{dt} = 2\pi r\frac{dr}{dt}$, the value of $\frac{dA}{dt}$ depends on r.

38. A. $v = s^3$

$$dv = 3s^2 ds$$
$$24 = 3s^2(2)$$
$$s = 2 \text{ in}$$

39. E. $sA = 6s^2$

$$dsA = 12sds$$
$$12 = 12sds$$
$$ds = \frac{1}{s}$$
$$V = s^3$$
$$dV = 3s^2 ds = 3s^2\frac{1}{s}$$
$$24 = 3s$$
$$s = 8 \text{ in}$$

40. C. $\frac{x}{y}\frac{dx}{dt} = \frac{dy}{dt}$

$$\frac{0.6}{0.8}3 = \frac{dy}{dt}$$

$\frac{dy}{dt} = 2.25$, but it is negative because y is decreasing.

$$\frac{dy}{dt} = -2.25.$$

41. B. $v = \pi r^2 h$

$$sA = 2\pi rh$$
$$dv = \pi r^2 dh$$
$$dsA = 2\pi h dr$$
$$dv = dsA$$
$$\pi r^2 dh = 2\pi h dr$$
$$\frac{dh}{h} = 2\frac{dr}{r^2}$$
$$\frac{2}{100} = 2\frac{dr}{(1)^2}$$
$$dr = .01\frac{\text{cm}}{\text{sec}}$$

42. (a) $\frac{dc}{dt} = \frac{d}{dt}(x^3 - 6x^2 + 15x)$

$$= (3x^2 - 12x + 15)\frac{dx}{dt}$$
$$= [3(2)^2 - 12(2) + 15](0.1)$$
$$= 0.3$$

$$\frac{dr}{dt} = \frac{d}{dt}(9x) = 9\frac{dx}{dt} = 9(0.1) = 0.9$$

$$\frac{dp}{dt} = \frac{dr}{dt} - \frac{dc}{dt} = 0.9 - 0.3 = 0.6$$

(b) $\frac{dc}{dt} = \frac{d}{dt}\left(x^3 - 6x^2 + \frac{45}{x}\right)$

$$= \left(3x^2 - 12x - \frac{45}{x^2}\right)\frac{dx}{dt}$$
$$= \left[3(1.5)^2 - 12(1.5) - \frac{45}{1.5^2}\right](0.05)$$
$$= -1.5625$$

42. Continued

(b) $\frac{dr}{dt} = \frac{d}{dt}(70x) = 70\frac{dx}{dt} = 70(0.05) = 3.5$

$\frac{dp}{dt} = \frac{dr}{dt} - \frac{dc}{dt} = 3.5 - (-1.5625) = 5.0625$

43. (a) Note that the level of the coffee in the cone is not needed until part (b).

Step 1:

V_1 = volume of coffee in pot

y = depth of coffee in pot

Step 2:

$\frac{dV_1}{dt} = 10 \text{ in}^3/\text{min}$

Step 3:

We want to find the value of $\frac{dy}{dt}$.

Step 4:

$V_1 = 9\pi y$

Step 5:

$\frac{dV_1}{dt} = 9\pi\frac{dy}{dt}$

Step 6:

$10 = 9\pi\frac{dy}{dt}$

$\frac{dy}{dt} = \frac{10}{9\pi} \approx 0.354$ in./min

The level in the pot is increasing at the rate of approximately 0.354 in./min.

(b) Step 1:

V_2 = volume of coffee in filter

r = radius of surface of coffee in filter

h = depth of coffee in filter

Step 2:

At the instant in question, $\frac{dV_2}{dt} = -10 \text{ in}^3/\text{min}$ and $h = 5$ in.

Step 3:

We want to find $-\frac{dh}{dt}$.

Step 4:

Note that $\frac{r}{h} = \frac{3}{6}$, so $r = \frac{h}{2}$.

Then $V_2 = \frac{1}{3}\pi r^2 h = \frac{\pi h^3}{12}$.

Step 5:

$\frac{dV_2}{dt} = \frac{\pi h^2}{4}\frac{dh}{dt}$

Step 6:

$-10 = \frac{\pi(5)^2}{4}\frac{dh}{dt}$

$\frac{dh}{dt} = -\frac{8}{5\pi}$ in./min

Note that $\frac{dh}{dt} < 0$, so the rate at which the level is *falling* is positive. The level in the come is falling at the rate of $\frac{8}{5\pi} \approx 0.509$ in./min.

44. Step 1:

Q = rate of CO_2 exhalation (mL/min)

D = difference between CO_2 concentration in blood pumped to the lungs and CO_2 concentration in blood returning from the lungs (mL/L)

y = cardiac output

Step 2:

At the instant in question, $Q = 233$ mL/min, $D = 41$ mL/L, $\frac{dD}{dt} = -2$ (mL/L)/min, and $\frac{dQ}{dt} = 0 \text{ mL/min}^2$.

Step 3:

We want to find the value of $\frac{dy}{dt}$.

Step 4:

$y = \frac{Q}{D}$

Step 5:

$\frac{dy}{dt} = \frac{D\frac{dQ}{dt} - Q\frac{dD}{dt}}{D^2}$

Step 6:

$\frac{dy}{dt} = \frac{(41)(0) - (233)(-2)}{(41)^2} = \frac{466}{1681} \approx 0.277 \text{ L/min}^2$

The cardiac output is increasing at the rate of approximately 0.277 L/min^{-2}.

45. (a) The point being plotted would correspond to a point on the edge of the wheel as the wheel turns.

(b) One possible answer is $\theta = 16\pi t$, where t is in seconds. (An arbitrary constant may be added to this expression, and we have assumed counterclockwise motion.)

45. Continued

(c) In general, assuming counterclockwise motion:

$$\frac{dx}{dt} = -2\sin\theta\frac{d\theta}{dt} = -2(\sin\theta)(16\pi) = -32\pi\sin\theta$$

$$\frac{dy}{dt} = 2\cos\theta\frac{d\theta}{dt} = 2(\cos\theta)(16\pi) = 32\pi\cos\theta$$

At $\theta = \frac{\pi}{4}$:

$$\frac{dx}{dt} = -32\pi\sin\frac{\pi}{4} = -16\pi(\sqrt{2}) \approx -71.086 \text{ ft/sec}$$

$$\frac{dy}{dt} = 32\pi\cos\frac{\pi}{4} = 16\pi(\sqrt{2}) \approx 71.086 \text{ ft/sec}$$

At $\theta = \frac{\pi}{2}$:

$$\frac{dx}{dt} = -32\pi\sin\frac{\pi}{2} = -32\pi \approx -100.531 \text{ ft/sec}$$

$$\frac{dy}{dt} = 32\pi\cos\frac{\pi}{2} = 0 \text{ ft/sec}$$

At $\theta = \pi$:

$$\frac{dx}{dt} = -32\pi\sin\pi = 0 \text{ ft/sec}$$

$$\frac{dy}{dt} = 32\pi\cos\pi = -32\pi \approx -100.531 \text{ ft/sec}$$

46. (a) One possible answer:

$x = 30\cos\theta,\ y = 40 + 30\sin\theta$

(b) Since the ferris wheel makes one revolution every 10 sec, we may let $\theta = 0.2\pi t$ and we may write $x = 30\cos 0.2\pi t,\ y = 40 + 30\sin 0.2\pi t$. (This assumes that the ferris wheel revolves counterclockwise.)

In general:

$$\frac{dx}{dt} = -30(\sin 0.2\pi t)(0.2\pi) = -6\pi\sin 0.2\pi t$$

$$\frac{dy}{dt} = 30(\cos 0.2\pi t)(0.2\pi) = 6\pi\cos 0.2\pi t$$

At $t = 5$:

$$\frac{dx}{dt} = -6\pi\sin\pi = 0 \text{ ft/sec}$$

$$\frac{dy}{dt} = 6\pi\cos\pi = 6\pi(-1) \approx -18.850 \text{ ft/sec}$$

At $t = 8$:

$$\frac{dx}{dt} = -6\pi\sin 1.6\pi \approx 17.927 \text{ ft/sec}$$

$$\frac{dy}{dt} = 6\pi\cos 1.6\pi \approx 5.825 \text{ ft/sec}$$

47. (a) $\frac{dy}{dt} = \frac{d}{dt}(uv) = u\frac{dv}{dt} + v\frac{du}{dt}$

$= u(0.05v) + v(0.04u)$

$= 0.09uv$

$= 0.09y$

Since $\frac{dy}{dt} = 0.09y$, the rate of growth of total production is 9% per year.

(b) $\frac{dy}{dt} = \frac{d}{dt}(uv) = u\frac{dv}{dt} + v\frac{du}{dt}$

$= u(0.03v) + v(-0.02u)$

$= 0.01uv$

$= 0.01y$

The total production is increasing at the rate of 1% per year.

Quick Quiz Sections 4.4–4.6

1. B. $x_{n+1} = x_n - \frac{f(x)}{f'(x)}$

$f(x) = x^3 + 2x - 1$

$f'(x) = 3x^2 + 2$

$$x_2 = 1 - \frac{(1)^3 + 2(1) - 1}{3(1)^2 + 2} = \frac{3}{5}$$

$$x_3 = \frac{3}{5} - \frac{\left(\frac{3}{5}\right)^3 + 2\left(\frac{3}{5}\right) - 1}{3\left(\frac{3}{5}\right)^2 + 2} = 0.465$$

2. B. $z^2 = x^2 + y^2$

$z = \sqrt{4^2 + 3^2} = 5$

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$5 = 4\left(3\frac{dy}{dt}\right) + 3\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{3}$$

$$\frac{dx}{dt} = 3\frac{dy}{dt} = 3\left(\frac{1}{3}\right) = 1$$

3. A. $x(t) = 70$

$y(t) = 60t$

$z(t) = ((60t)^2 + 70^2)^{1/2}$

$$\frac{dz}{dt} = \frac{1}{2}(3600t^2 + 4900)^{-1/2}(7200t)$$

$$\frac{dz}{dt} = \frac{7200(4)}{2(3600(4)^2 + 4900)^{1/2}}$$

$$\frac{dz}{dt} = 57.6$$

4. (a) $f(x) = \sqrt{x}$

$x = 25$

$$f'(25) = \frac{1}{2}(25)^{-1/2} = \frac{1}{10}$$

$$\sqrt{26} = 5 + \frac{1}{10}(26 - 25) = 5.1$$